CISC-102 Fall 2016

Quiz 2

October 18, 2016

Student ID: SOLUTIONS

Read the questions carefully. Please clearly state any assumptions that you make that are not explicitly stated in the question. Please answer all questions in the space provided. Use the back of pages for scratch work. There are 4 pages to this quiz. Note that (x) denotes the question is worth x points.

CALCULATORS ARE NOT PERMITTED.

1. Consider a function f from the set A to B where A $= \{x : x \in \mathbb{R}, 1 \le x \le 10\}$, and B $= \{x : x \in \mathbb{N}, 1 \le x \le 10\}$, and the mapping is defined as

   $f(x)$ = greatest element of B that is less than or equal to x. This is a special case of the *floor* function defined over a reduced domain and range. The function simply takes a real number in the closed interval $[1 \dots 10]$ and rounds down to the closest integer. For example $f(2.3456) = 2$, $f(1.99999) = 1$, $f(7) = 7$.

   (a) (2) Is this function one-to one? Explain your answer.

   No, for example $f(2.1) = f(2.2) = 2$

   (b) (2) Is this function onto? Explain your answer.

   Yes, every element of B is an image of the function.

   (c) (2) Is this function a bijection? Explain your answer.

   A bijection is a function that is one-to-one and onto. f is not one-to-one so its not a bijection.

2. Consider a relation Abs that is a subset of $\mathbb{Z} \times \mathbb{Z}$, such $\{(a,b) : a,b \in \mathbb{Z}, b = |a|\}$. This is the absolute value function, but since functions are a special case of relations we can consider Abs as a relation.

   (a) (2) Is this relation reflexive? Explain your answer?

   No, for example $(-1,-1) \notin Abs$.

   (b) (2) Is this relation symmetric? Explain your answer?

   No, for example $(-1,1) \in Abs$
   but $(1,-1) \notin Abs$.

   (c) (2) Is this relation antisymmetric? Explain your answer?

   Yes, because whenever we have
   (a,b) & $(b,a) \in Abs$, $a=b$.

   (d) (2) Is this relation transitive? Explain your answer?

   Yes, because whenever we have
   (a,b), $(b,c) \in Abs$, we also have $(a,c) \in Abs$.

   (e) (2) Is this relation a partial order? Explain your answer?

   No, a partial order is reflexive, antisymmetric,
   and transitive, Abs is not reflexive.

3. (6) Consider the relation $R = \{(a,b) \in \mathbb{N}^2 : |a-b| \leq 2\}$. For example $(1,3) \in R$ and $(3,1) \in R$ but $(1,4) \notin R$. Is R an equivalence relation? Explain your answer.

A relation is an equivalence relation if it is reflexive, symmetric, and transitive. R is not transitive because $(1,2)$ & $(2,4) \in R$ but $(1,4) \notin R$, So R is not an equivalence relation.

4. (6) Prove using mathematical induction that the sum of the first n natural numbers is less than or equal to n^2. This can also be stated as:

Prove that the proposition P(n),

$$\sum_{i=1}^{n} i \leq n^2$$

is true for all $n \in \mathbb{N}$.

Proof:

Base: $1 \leq 1^2$

Ind. Hyp.: Assume $\sum_{i=1}^{k} i \leq k^2$ for $k \geq 1$.

Ind. Step:

$$\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + k+1$$

$$\leq k^2 + k+1 \quad \text{(Ind. Hyp.)}$$

$$\leq k^2 + 2k + 1$$

$$= (k+1)^2$$

So by the principle of math induction we conclude that $p(n)$ is true for all $n \in \mathbb{N}$. ∎

5. (6) Show using the Division Algorithm theorem, that if $a|b$ and $a|c$ then $a|b+c$.

Proof: By the Div. Alg. theorem we have $b = a p_a$ and $c = a p_c$ $p_a, p_c \in \mathbb{Z}$,

So $b + c = a(p_a + p_c)$

which implies that $a \mid b + c$. $\boxed{}$

6. (6) Use mathematical induction to prove P(n), the proposition that $2|(n^2 + n)$ for all $n \in \mathbb{N}$.

Proof:

Base: $2 \mid 1^2 + 1$

Ind Hyp.: Assume $2 \mid (K^2 + K)$

Ind. Step: Consider,

$$(K+1)^2 + K + 1 = K^2 + 2K + 1 + K + 1$$
$$= K^2 + K + 2(K+1).$$

$2 \mid K^2 + K$ by the ind. hyp.

$2 \mid 2(K+1)$

therefore $2 \mid K^2 + K + 2(K+1)$ (see Q.5)

By the principle of math ind. we conclude that $P(n)$ is true for all $n \in \mathbb{N}$. $\boxed{}$