CISC-102 Fall 2016

Quiz 3

November 8, 2016

Student ID: Solutions

Read the questions carefully. Please clearly state any assumptions that you make that are not explicitly stated in the question. Please answer all questions in the space provided. Use the back of pages for scratch work. There are 5 pages to this quiz. Note that (x) denotes the question is worth x points.

CALCULATORS ARE NOT PERMITTED.

1. Let a = 17, and b = 13.

   (a) (4) Find $g = \gcd(a, b)$. Show the steps used by Euclid's algorithm to find $\gcd(a, b)$.

   $17 = (1)13 + 4$

   $13 = (3)4 + 1$

   $4 = (4)1 + 0$

   so $\gcd(17, 13) = 1$

   (b) (4) Verify that there are integers p and q such that $\gcd(17, 13) = p \times 17 + q \times 13$.

   $1 = 13 - (3)4$

   $= 13 - (3)\boxed{17 - 13}$

   $= (4)13 - (3)17$

2. (4) Let $g = \gcd(m, n)$. Show that $m \equiv n (\mathrm{mod}\ g)$.

By defn $m \equiv n \ (\mathrm{mod}\ g)$ implies $g \mid (m-n)$

$g \mid m$ and $g \mid n$ because $g = \gcd(m,n)$

therefore $g \mid m-n$. $\boxed{E}$

3. (4) In how many different ways can the letters SASKATOON be rearranged?

$$\frac{9!}{2!\,2!\,2!}$$

4. (6) Use the second form of induction, that is, strong induction, to prove that any integer value greater than or equal to 2, can be written as $2a + 3b$, where a, b are non-negative integers, that is $a, b \in \mathbb{Z}, a, b \geq 0$. (Hint: Consider separate cases.)

Base $2 = 2(1) + 3(0)$
$3 = 2(0) + 3(1)$

Ind. hyp. Assume j, $2 \leq j \leq k$ can be written as $2a + 3b$.

Ind. step Consider $k+1$.
$k = 3a + 3b$ by the ind. hyp.

case: $a > 0$
$k+1 = 2(a-1) + 3(b+1)$

case $a = 0$, $b > 0$
$k+1 = 2(a+2) + 3(b-1)$.

5. (4) In this question we consider a six card poker hand. A six card poker hand is called *two pair* if it consists of two distinct pairs of the same value and fifth and sixth cards with values different from the first two and different from each other. For example:

$$2\heartsuit, 2\diamondsuit, 3\clubsuit, 3\spadesuit 6\heartsuit 7\heartsuit \text{ and } 2\heartsuit, 2\clubsuit, 3\diamondsuit, 5\diamondsuit, 6\diamondsuit 6\heartsuit.$$

How many six card hands (unordered) can be drawn from a 52 card deck (13 values $\times$ 4 suits) so that we get two pair.

$$\binom{13}{2}\binom{11}{2}\binom{4}{2}^2\binom{4}{1}^2$$

6. You have 20 identical dollar coins to distribute amongst 5 people.

(a) (4)In how many ways can the coins be distributed?

This problem is equivalent to counting the number of binary strings of length 20+5-1, using 20 0s and 4 1's. Therefore we have

$$\frac{24!}{4!\,20!} = \binom{24}{4} = \binom{24}{20}$$

ways to distribute the coins.

(b) (4) In how many ways can the coins be distributed so that each person gets at least 2 coins?

First give 2 coins to each person leaving 10 that are not distributed. Using a technique similar to 4(a) we have:

$$\frac{(10+5-1)!}{10!\,4!} = \binom{14}{4} = \binom{14}{10}$$

ways to distribute the coins.

7. Consider the sum:

$$S_n = \sum_{i=0}^{n} \binom{n}{i} 3^{n-i}(-1)^i.$$

(a) (2) Verify that $S_2 = 4$.

$$S_2 = \binom{2}{0} 3^2 (-1)^0 + \binom{2}{1} 3 (-1)^1 + \binom{2}{2} 3^0 (-1)^2$$
$$= 9 - 6 + 1 = 4$$

(b) (4) Use the binomial theorem to prove that $S_n = 2^n$.

The binomial theorem is:

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} (x)^{n-i} (y)^i$$

letting $x = 3$ and $y = -1$ we get:

$$(3-1)^n = 2^n = \sum_{i=0}^{n} \binom{n}{i} 3^{(n-i)} (-1^i)$$