Problems

1. Illustrate DeMorgan’s Law \((A \cap B)^c = A^c \cup B^c\) using Venn diagrams.

Figure 1: \((A \cap B)\) is shown in (a), and (c) and (d) illustrate \(B^c\) and \(A^c\) respectively. Finally (b) shows that \((A \cap B)^c = A^c \cup B^c\)

2. Write the dual of each of the following set equations.

   (a) \(A \cup (A \cap B) = A\)
       \(A \cap (A \cup B) = A\)
   (b) \((A \cup U) \cap (A \cap \emptyset) = \emptyset\)
       \((A \cap \emptyset) \cup (A \cup U) = U\)
   (c) \(A^c \cup B^c \cup C^c = (A \cap B \cap C)^c\)
       \(A^c \cap B^c \cap C^c = (A \cup B \cup C)^c\)

3. Let \(A_i = \{1, 2, 3, \ldots, i\}\) for all \(i \in \mathbb{N}\). For example \(A_4 = \{1, 2, 3, 4\}\).
   What are the elements of the set:
(a) What are the elements of the set $\bigcup_{i=1}^{n} A_i$?

$$\bigcup_{i=1}^{n} A_i = \{1, 2, \ldots, n\}$$

(b) What are the elements of the set $\bigcap_{i=1}^{n} A_i$?

$$\bigcap_{i=1}^{n} A_i = \{1\}$$

Figure 2: $A \subseteq B$ is shown on the left, and $A \subseteq B \subseteq C$ is shown on the right.

4. Observe that $A \subseteq B$ has the same meaning as $A \cap B = A$. Draw a Venn diagram to illustrate this fact.

See Figure 2. If $A \subseteq B$ then every element $x \in A$ is also an element in $B$, which in turn implies that $A \cap B = A$.

5. Use a Venn diagram to show that if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

See Figure 2. $A \subseteq B$ implies that every element of $A$ is also in $B$, $x \in A$ implies $x \in B$. Similarly $B \subseteq C$ implies that implies that every element of $B$ is also in $C$, $y \in B$ implies $y \in C$. Thus $A \subseteq C$.

6. Use the Principle of Exclusion and Inclusion to show that $|A \cup B| + |A \cap B| = |A| + |B|$.

(It may help your understanding if you first explore an example such as $A = \{1,2,3\}$ and $B = \{3,4\}$).

By the Principle of Inclusion Exclusion we have $|A| + |B| - |A \cap B| = |A \cup B|$. These quantities are just non-negative integers so if we add $|A \cap B|$ to the right and left side of the equation, we get the desired result.

7. What are the cardinalities of the following sets?

   (a) $A = \{\text{winter, spring, summer, fall}\}$. $|A| = 4$.

   (b) $B = \{x : x \in \mathbb{Z}, 0 < x < 7\}$. $|B| = 6$. 

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(c) \( P(B) \), that is, the power set of \( B \). \(|P(B)| = 2^6 = 64 \).

(d) \( C = \{ \ x : x \in \mathbb{N}, x \text{ is even} \} \) This set has infinitely many elements.

8. Suppose that we have a sample of 100 students at Queen’s who take at least one of the following language courses, French-101, Spanish-101, German-101. Also suppose that 65 take French-101, 45 take German-101, 42 take Spanish 101, 20 take French-101 and German-101, 25 take French-101 and Spanish-101, and 15 take German-101 and Spanish-101.

(a) How many students take all three language courses?

Let \( F, S, \) and \( G \) denote the sets of students taking French Spanish and German respectively. The Principle of Inclusion and Exclusion tells us that

\[
|F \cup S \cup G| = |F| + |S| + |G| - |F \cap S| - |S \cap G| - |F \cap G| + |F \cap S \cap G|
\]

The problem statement gives us values for each quantity in the equation except for \(|F \cap S \cap G|\). We can now simply fill in the numbers and solve for \(|F \cap S \cap G|\), as follows:

\[
100 = 65 + 42 + 45 - 25 - 15 - 20 + |F \cap S \cap G|\]

So we conclude that \(|F \cap S \cap G| = 8\).

(b) Draw a Venn diagram representing these 100 students and fill in the regions with the correct number.

(c) How many students take exactly 1 of these courses? Using the Venn diagram we can deduce that \(28+10+18 = 56\) students take exactly one of the language courses.

(d) How many students take exactly 2 of these courses? Using the Venn diagram we can deduce that \(17 +12 +7 = 36\) students take exactly two courses.

9. Let \( S=\{a,b,c,d,e,f,g\} \). Determine which of the following are partitions of \( S \):

(a) \( P_1 = \{\{a,c,e\},\{b\},\{d,g\}\} \) No, because \( f \) is missing from the union of the sets.

(b) \( P_2 = \{\{a,b,e,g\},\{c\},\{d,f\}\} \) Yes. The union of the sets is \( S \), and the pairwise intersections of the sets are empty.

(c) \( P_3 = \{\{a,e,g\},\{c,d\},\{b,e,f\}\} \) No, because the intersection of \( \{a, e, g\} \cap \{b, e, f\} \) is not empty.

(d) \( P_4= \{\{a,b,c,d,e,f,g\}\} \) Yes, this is technically a partition, but a very uninteresting one.
10. Recall that the union operation is associative, that is \( A \cup (B \cup C) = (A \cup B) \cup C \). Show that the relative complement set operation is not associative, that is, \( A \setminus (B \setminus C) \neq (A \setminus B) \setminus C \), is incorrect for some sets A, B, C. (Note if relative complement is associative then the equation must be true for all sets A, B, C.)

Let \( A = \{1,2,3\} \), \( B = \{1,2\} \) and \( C = \{2,3\} \). \( A \setminus (B \setminus C) = \{2,3\} \) and \( (A \setminus B) \setminus C = \emptyset \).