CISC-102 FALL 2016

HOMEWORK 3
SOLUTIONS

(1) Prove using mathematical induction that the sum of the first n natural numbers is equal to $\frac{n(n+1)}{2}$. This can also be stated as:

Prove that the proposition P(n),

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

is true for all $n \in \mathbb{N}$.

Base: for n =1, $1 = \frac{1(1+1)}{2}$

Induction hypothesis: Assume that P(k) is true, that is:

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}.$$

for $k \geq 1$.

Induction step: The goal is to show that P$(k+1)$ is true, that is:

$$\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}.$$

Consider the sum

$$\begin{aligned}
\sum_{i=1}^{k+1} i =& \sum_{i=1}^{k} i + (k+1)\text{(arithmetic)}\\
=& \frac{k(k+1)}{2} + (k+1)\text{(Use the induction hypothesis)}\\
=& \frac{k^2+k+2k+2}{2}\text{(get common denominator and add)}\\
=& \frac{k^2+3k+2}{2}\text{(add k + 2k)}\\
=& \frac{(k+1)(k+2)}{2}\text{(factor to arrive at goal)}
\end{aligned}$$

We have shown that P(k) true implies that P$(k+1)$ is true so by the principle of mathematical induction we conclude that P(n) is true for all $n \in \mathbb{N}$. $\square$

(2) Prove using mathematical induction that the proposition P(n),

$$\sum_{i=1}^{n} \frac{1}{2^i} = 1 - \frac{1}{2^n}$$

is true for all $n \in \mathbb{N}$.

Base: $\frac{1}{2} = 1 - \frac{1}{2}$

Induction hypothesis: Assume that P(k) is true, that is:

$$\sum_{i=1}^{k} \frac{1}{2^i} = 1 - \frac{1}{2^k}.$$

Induction step: The goal is to show that P($k+1$) is true that is,

$$\sum_{i=1}^{k+1} \frac{1}{2^i} = 1 - \frac{1}{2^{k+1}}.$$

Consider the sum:

$$\begin{aligned}\sum_{i=1}^{k+1}\frac{1}{2^i} =& \sum_{i=1}^{k}\frac{1}{2^i} + \frac{1}{2^{k+1}}\text{(arithmetic)}\\ =&1 - \frac{1}{2^k} + \frac{1}{2^{k+1}}\text{(Use the induction hypothesis)}\\ =&1 + \frac{-2+1}{2^{k+1}}\text{(get common denominator)}\\ =&1 - \frac{1}{2^{k+1}}\text{(add to arrive at goal)}\end{aligned}$$

We have shown that P(k) true implies that P($k+1$) is true so by the principle of mathematical induction we conclude that P(n) is true for all $n \in \mathbb{N}$. □

(3) Prove using mathematical induction that the proposition P(n)

$$n! \leq n^n$$

is true for all $n \in \mathbb{N}$.

Base: for n =1, $1! = 1 = 1^1$

Induction hypothesis: Assume that P(k) is true, that is:

$$k! \leq k^k$$

for $k \geq 1$.

Induction step: The goal is to show that P$(k+1)$ is true, that is:

$$(k+1)! \leq (k+1)^{k+1}.$$

We have:

$$\begin{aligned}(k+1)! =& k!(k+1) \text{(Definition of factorial)}\\ \leq& k^k(k+1) \text{(Use the induction hypothesis)}\\ \leq& (k+1)^k(k+1) \text{(because } k \leq k+1)\\ =& (k+1)^{k+1} \text{(multiply)}\end{aligned}$$

We have shown that P(k) true implies that P$(k+1)$ is true so by the principle of mathematical induction we conclude that P(n) is true for all $n \in \mathbb{N}$. $\square$

(4) Let S be a set of n elements, such that $a \in S$. Show that there are the same number of subsets of S that do contain a as there are subsets of S that do not contain a.

Let $S' = S \setminus \{a\}$. Observe that P(S') that is all subsets of S' are exactly the subsets of S that do not contain the element a. Now consider the set (of sets) S'' obtained by performing $s \cup \{a\}$ for every set $s \in P(S')$. This gives us all subsets of S that include the element a. It should be obvious now that $|S'| = |S''|$ and that $S' \cup S'' = P(S)$.