(1) Determine whether the mappings given below where \( f : \mathbb{R} \to \mathbb{R} \) are or are not functions, and explain your decision.

(a) \( f(x) = 1/x \)

\( f(x) = 1/x \) is not a function from \( \mathbb{R} \) to \( \mathbb{R} \) because \( 1/x \) is not defined for \( x = 0 \).

\( f(x) = 1/x \) is a functions from \( \mathbb{R} \setminus \{0\} \) to \( \mathbb{R} \).

(b) \( f(x) = \sqrt{x} \)

\( f(x) = \sqrt{x} \) is not a function from \( \mathbb{R} \) to \( \mathbb{R} \) because \( \sqrt{x} \) is not a real number for \( x < 0 \). Furthermore, \( \sqrt{x} \) has a positive and negative value for \( x \in \mathbb{R}, x > 0 \). We could salvage this by defining the set \( \mathbb{R}^+ = \{x : x \in \mathbb{R}, x \geq 0\} \), and consider a function from \( \mathbb{R}^+ \) to \( \mathbb{R}^+ \) defined as \( f(x) = +\sqrt{x} \).

(c) \( f(x) = 3x - 3 \)

Multiplication by a constant and subtraction of a constant is closed under the reals. Therefore, \( f(x) = 3x - 3 \) is a function because \( 3x - 3 \) has a distinct image \( x \in \mathbb{R} \).

(2) Determine whether each of the following functions from \( \mathbb{R} \) to \( \mathbb{R} \) is a bijection, and explain your decision. HINT: Plotting these functions may help you with your decision.

(a) \( f(x) = 3x + 4 \)

\( f(x) = 3x + 4 \) is an onto function. Consider the equation \( y = 3x + 4 \). For every real valued \( y \) we can find a real valued \( x \), that is \( x = (y - 4)/3 \). \( f(x) = 3x + 4 \) is a one-to-one function because, if \( 3x_1 + 4 = 3x_2 + 4 \) then \( x_1 = x_2 \). Therefore we can conclude that \( f(x) = 3x + 4 \) is a bijection. Furthermore we have \( f^{-1}(x) = x/3 - 4 \).

(b) \( f(x) = -x^2 + 2 \)

\( f(x) = -x^2 + 2 \) is not a bijection. It is neither onto \( (f(x) \leq 2) \) nor one-to-one \( (f(x) = 0 \text{ for } x = +\sqrt{2} \text{ and for } x = -\sqrt{2}) \).

(c) \( f(x) = x^3 - x^2 \)

\( f(x) = x^3 - x^2 \) is not a bijection. It is not one-to-one because \( x^3 - x^2 = x^2(x-1) \) and is equal to 0 for \( x = 1 \), and \( x = 0 \).
(3) Consider the recursive function \( T(1) = 1, T(n) = T(n-1) + 1 \), for all \( n \geq 2 \).

(a) Use the recursive definition to obtain values \( T(2), T(3), \) and \( T(4) \).
\[
T(2) = 2, \quad T(3) = 3, \quad T(4) = 4.
\]

(b) Using the values that you obtained for \( T(2), T(3), \) and \( T(4) \), to guess the value of \( T(n) \), and then prove that it is correct using induction.

We guess that \( T(n) = n \), and prove this using mathematical induction.

Let \( P(n) \) denote the proposition that the recursive function \( T(n) \) as defined above has the closed form solution \( T(n) = n \).

\( P(n) \) is true for all \( n \in \mathbb{N} \).

**Proof:** We use mathematical induction.

**Base:** \( T(1) = 1 \) by definition.

**Induction Hypothesis:** Assume that \( P(k) \) is true for some \( k, k \geq 1 \), that is, \( T(k) = k \).

**Induction Step:** \( P(k+1) \) is the proposition that \( T(k+1) = k + 1 \), and we show that it is true using the induction hypothesis.

\[
T(k + 1) = T(k) + 1 \text{ (definition of } T(k+1)) \\
= k + 1 \text{ (induction hypothesis)}
\]

We have shown that \( P(k) \) true implies that \( P(k+1) \) is true so by the principle of mathematical induction we conclude that \( P(n) \) is true for all \( n \in \mathbb{N} \). \( \Box \)

(4) Consider the following relations on the set \( A = \{1, 2, 3\} \):

- \( R = \{(1, 1), (1, 2), (1, 3), (3, 3)\} \),
- \( S = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 3)\} \),
- \( T = \{(1, 1), (1, 2), (2, 2), (2, 3)\} \),
- \( A \times A \).

For each of these relations determine whether it is symmetric, antisymmetric, reflexive, or transitive.

\( S \) and \( A \times A \) are symmetric.
\( R \) and \( T \) are antisymmetric.
\( S \) and \( A \times A \) are reflexive.
\( R, S \) and \( A \times A \) are transitive.
Figure 1. (a) $\frac{1}{x}$ (b) $\sqrt{x}$ (c) $3x - 3$ (d) $3x + 4$ (e) $-x^2 + 2$ (f) $x^3 - x^2$