CISC-102 FALL 2016

HOMEWORK 4 SOLUTIONS

(1) Determine whether the mappings given below where $f : \mathbb{R} \mapsto \mathbb{R}$ are or are not functions, and explain your decision.

(a) $f(x) = 1/x$
$f(x) = 1/x$ is not a function from $\mathbb{R}$ to $\mathbb{R}$ because $1/x$ is not defined for $x = 0$. $f(x) = 1/x$ is a functions from $\mathbb{R} \setminus \{0\}$ to $\mathbb{R}$.

(b) $f(x) = \sqrt{x}$
$f(x) = \sqrt{x}$ is not a function from $\mathbb{R}$ to $\mathbb{R}$ because $\sqrt{x}$ is not a real number for $x < 0$. Furthermore, $\sqrt{x}$ has a positive and negative value for $x \in \mathbb{R}, x > 0$. We could salvage this by defining the set $\mathbb{R}^+ = \{x : x \in \mathbb{R}, x \geq 0\}$, and consider a function from $\mathbb{R}^+$ to $\mathbb{R}^+$ defined as $f(x) = +\sqrt{x}$.

(c) $f(x) = 3x - 3$
Multiplication by a constant and subtraction of a constant is closed under the reals. Therefore, $f(x) = 3x - 3$ is a function because $3x - 3$ has a distinct image $x \in \mathbb{R}$.

(2) Determine whether each of the following functions from $\mathbb{R}$ to $\mathbb{R}$ is a bijection, and explain your decision. HINT: Plotting these functions may help you with your decision.

(a) $f(x) = 3x + 4$
$f(x) = 3x+4$ is an onto function. Consider the equation $y = 3x+4$. For every real valued y we can find a real valued x, that is $x = y/3 - 4$. $f(x) = 3x + 4$ is a one-to-one function because, if $3x_1 + 4 = 3x_2 + 4$ then $x_1 = x_2$. Therefore we can conclude that $f(x) = 3x + 4$ is a bijection. Furthermore we have $f^{-1}(x) = x/3 - 4$

(b) $f(x) = -x^2 + 2$
$f(x) = -x^2 + 2$ is not a bijection. It is neither onto ($f(x) \leq 2$) nor one-to-one ($f(x) = 0$ for $x = +\sqrt{2}$ and for $x = -\sqrt{2}$).

(c) $f(x) = x^3 - x^2$
$f(x) = x^3-x^2$ is not a bijection. It is not one-to-one because $x^3-x^2 = x^2(x-1)$ and is equal to 0 for $x = 1$, and $x = 0$.

(3) Consider the recursive function $T(1) = 1, T(n) = T(n-1) + 1$, for all $n \geq 2$.

(a) Use the recursive definition to obtain values T(2), T(3), and T(4).
T(2) = 2, T(3) = 3, T(4) = 4.

(b) Using the values that you obtained for T(2), T(3), and T(4), to guess the value of T(n), and then prove that it is correct using induction.

We guess that T$(n) = n$, and prove this using mathematical induction.

Let P(n) denote the proposition that the recursive function T(n) as defined above has the closed form solution T$(n) = n$.

P(n) is true for all $n \in \mathbb{N}$.

Proof: We use mathematical induction.

Base: T(1) = 1 by definition.

Induction Hypothesis: Assume that P(k) is true for some $k, k \geq 1$, that is, T$(k) = k$.

Induction Step: P$(k+1)$ is the proposition that T$(k+1) = k+1$, and we show that it is true using the induction hypothesis.

$$
\begin{aligned}
T(k+1) &= T(k) + 1 (\text{defintion of T(k+1)}) \\
&= k + 1 (\text{induction hypothesis})
\end{aligned}
$$

We have shown that P(k) true implies that P$(k+1)$ is true so by the principle of mathematical induction we conclude that P(n) is true for all $n \in \mathbb{N}$. □

(4) Consider the following relations on the set $A = \{1, 2, 3\}$:

- $R = \{(1,1), (1,2), (1,3), (3,3)\}$,
- $S = \{(1,1), (1,2), (2,1), (2,2), (3,3)\}$,
- $T = \{(1,1), (1,2), (2,2), (2,3)\}$,
- $A \times A$.

For each of these relations determine whether it is symmetric, antisymmetric, reflexive, or transitive.

S and A × A are symmetric.

R and T are antisymmetric.

S and A × A are reflexive.

R, S and A × A are transitive.

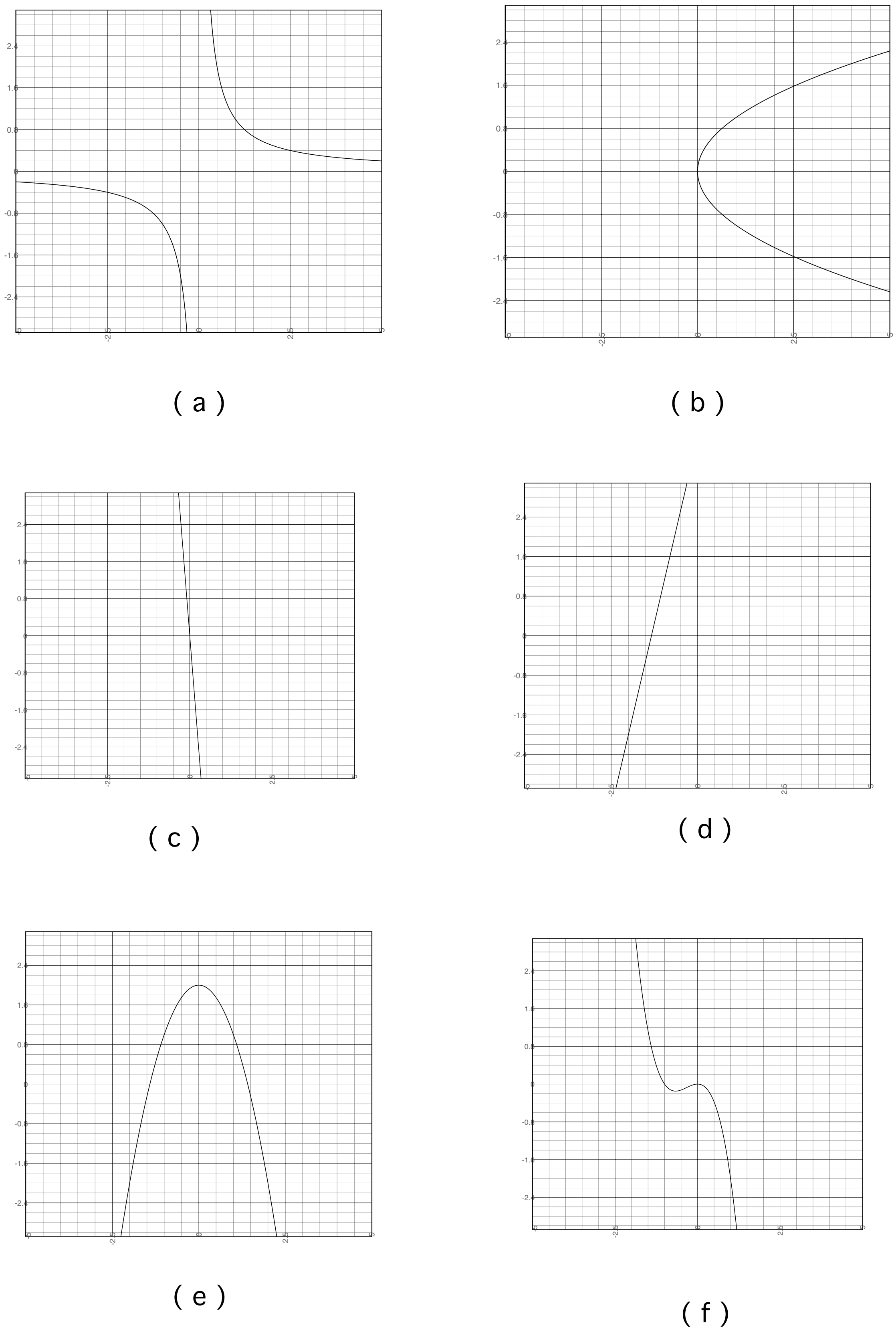

FIGURE 1. (a) $1/x$ (b) $\sqrt{x}$ (c) $3x - 3$ (d) $3x + 4$ (e) $-x^2 + 2$ (f) $x^3 - x^2$