CISC-102 FALL 2016

HOMEWORK 5 SOLUTIONS

(1) Evaluate

(a) $|3-7| = |-4| = 4$

(b) $|1-4| - |2-9| = |-3| - |-7| = -4$

(c) $|-6-2| - |2-6| = |-8| - |-4| = 4$

(2) Find the quotient q and remainder r, as given by the Division Algorithm theorem for the following examples.

Recall we want to find $r, 0 \leq r < |b|$, such that $a = qb + r$, where all values are integers.

(a) $a = 500, b = 17$.
$500 = 29 \times 17 + 7$ so $r = 7$.

(b) $a = -500, b = 17$.
$-500 = -30 \times 17 + 10$ so $r = 10$.

(c) $a = 500, b = -17$.
$500 = -29 \times -17 + 7$ so $r = 7$

(d) $a = -500, b = -17$
$-500 = 30 \times -17 + 10$ so $r = 10$

(3) Show that $c|0$, for all $c \in \mathbb{Z}, c \neq 0$.

Recall the definition of divisibility:

If $c = \frac{b}{a}$ is an integer, or alternately if c is an integer such that $b = ca$ then we say that a divides b or equivalently, b is divisible by a, and this is written $a|b$.

Since $\frac{0}{c} = 0$ for all $c \in \mathbb{Z}, c \neq 0$, and 0 is an integer we have shown that every integer c divides 0. Note: $\frac{0}{0}$ is undefined.

(4) Let $a, b, c \in \mathbb{Z}$ such that $c|a$ and $c|b$. Let r be the remainder of the division of b by a, that is there is a $q \in \mathbb{Z}$ such that $b = qa + r, 0 \leq r < |b|$. Show that under these condition we have $c|r$.

Since $c|a$ and $c|b$ we can write:

$$(1) a = cp_a \text{ and } b = cp_b, \text{ such that } p_a, p_b \in \mathbb{Z}.$$

So we can rewrite $b = qa + r$ as:

$$cp_b = qcp_a + r$$

and this simplifies to:

$$c(p_b - qp_a) = r$$

Since $p_b - qp_a$ is an integer we can conclude that $c|r$.

(5) Let $a, b \in \mathbb{Z}$ such that $2|a$. (In other words a is even.) Show that $2|ab$.

This is just a special case of the divisibility theorem that states if $c|a$ then for any integer b, $c|ab$

(6) Let $a \in \mathbb{Z}$ show that $3|a(a+1)(a+2)$, that is the product of three consecutive integers is divisible by 3.

Observe that we can write $a = 3q+r$ where $r \in \{0, 1, 2\}$.

Case 0: If $r = 0$ a is divisible by 3 and since $(a+1)(a+2)$ is an integer it follows that $3|a(a+1)(a+2)$.

Case 1: If $r = 1$, add 2 to both sides of the equation $a = 3q+1$ to get $a+2 = 3q+3 = 3(q+1)$ thus $a+2$ is divisible by 3 and since $a(a+1)$ is an integer it follows that $3|a(a+1)(a+2)$.

Case 2: If $r = 2$, add 1 to both sides of the equation $a = 3q+1$ to get $a+1 = 3q+3 = 3(q+1)$ thus $a+1$ is divisible by 3 and since $a(a+2)$ is an integer it follows that $3|a(a+1)(a+2)$.

(7) Use induction to prove $n^3 + 2n$ is divisible by 3, for all $n \in \mathbb{N}, n \geq 1$.

Base: $3|1^3 + 2$

Induction Hypothesis: Assume that $k^3 + 2k$ is divisible by 3, for $k \geq 1$.

Induction Step: Goal: Show that $3|(k+1)^3+2(k+1)$ using the induction hypothesis.

We begin by manipulating the expression $(k+1)^3+2(k+1)$ as follows:

$$
\begin{aligned}
(k+1)^3 + 2(k+1) &= k^3 + 3k^2 + 3k + 1 + 2k + 2 \\
&= k^3 + 2k + 3(k^2 + k + 1)
\end{aligned}
$$

Observe that $3|k^3 + 2k$ by the induction hypothesis and $3|3(k^2 + k + 1)$. So $3|k^3 + 2k + 3(k^2 + k + 1)$.

Therefore by the principle of mathematical induction we conclude that n^3+2n is divisible by 3, for all $n \in \mathbb{N}, n \geq 1$. $\square$