(1) Evaluate

(a) $|3 - 7| = |-4| = 4$
(b) $|1 - 4| - |2 - 9| = | -3| - |-7| = -4$
(c) $|-6 - 2| - |2 - 6| = |-8| - |-4| = 4$

(2) Find the quotient $q$ and remainder $r$, as given by the Division Algorithm theorem for the following examples.

Recall we want to find $r, 0 \leq r < |b|$, such that $a = qb + r$, where all values are integers.

(a) $a = 500, b = 17.$
\[ 500 = 29 \times 17 + 7 \text{ so } r = 7. \]
(b) $a = -500, b = 17.$
\[ -500 = -30 \times 17 + 10 \text{ so } r = 10. \]
(c) $a = 500, b = -17.$
\[ 500 = -29 \times -17 + 7 \text{ so } r = 7 \]
(d) $a = -500, b = -17$
\[ -500 = 30 \times -17 + 10 \text{ so } r = 10 \]
(3) Show that $c | 0$, for all $c \in \mathbb{Z}, c \neq 0$.

Recall the definition of divisibility:

If $c = \frac{b}{a}$ is an integer, or alternately if $c$ is an integer such that $b = ca$ then we say that $a$ divides $b$ or equivalently, $b$ is divisible by $a$, and this is written $a | b$.

Since $\frac{0}{c} = 0$ for all $c \in \mathbb{Z}, c \neq 0$, and 0 is an integer we have shown that every integer $c$ divides 0. Note: $\frac{0}{0}$ is undefined.

(4) Let $a, b, c \in \mathbb{Z}$ such that $c | a$ and $c | b$. Let $r$ be the remainder of the division of $b$ by $a$, that is there is a $q \in \mathbb{Z}$ such that $b = qa + r, 0 \leq r < |b|$. Show that under these condition we have $c | r$.

Since $c | a$ and $c | b$ we can write:

\[(1) a = cp_a \text{ and } b = cp_b, \text{ such that } p_a, p_b \in \mathbb{Z}.\]

So we can rewrite $b = qa + r$ as:

\[cp_b = qcp_a + r\]

and this simplifies to:

\[c(p_b - qp_a) = r\]
Since $p_b - qp_a$ is an integer we can conclude that $c|r$.

(5) Let $a, b \in \mathbb{Z}$ such that $2|a$. (In other words $a$ is even.) Show that $2|ab$.

This is just a special case of the divisibility theorem that states if $c|a$ then for any integer $b$, $c|ab$.

(6) Let $a \in \mathbb{Z}$ show that $3|a(a+1)(a+2)$, that is the product of three consecutive integers is divisible by 3.

Observe that we can write $a = 3q + r$ where $r \in \{0, 1, 2\}$.

Case 0: If $r = 0$ a is divisible by 3 and since $(a+1)(a+2)$ is an integer it follows that $3|a(a + 1)(a + 2)$.

Case 1: If $r = 1$, add 2 to both sides of the equation $a = 3q + 1$ to get $a + 2 = 3q + 3 = 3(q + 1)$ thus $a + 2$ is divisible by 3 and since $a(a + 1)$ is an integer it follows that $3|a(a + 1)(a + 2)$.

Case 2: If $r = 2$, add 1 to both sides of the equation $a = 3q + 1$ to get $a + 1 = 3q + 3 = 3(q + 1)$ thus $a + 1$ is divisible by 3 and since $a(a + 2)$ is an integer it follows that $3|a(a + 1)(a + 2)$.

(7) Use induction to prove $n^3 + 2n$ is divisible by 3, for all $n \in \mathbb{N}, n \geq 1$.

**Base:** $3|1^3 + 2$
**Induction Hypothesis:** Assume that $k^3 + 2k$ is divisible by 3, for $k \geq 1$.

**Induction Step:** Goal: Show that $3|(k+1)^3 + 2(k+1)$ using the induction hypothesis.

We begin by manipulating the expression $(k+1)^3 + 2(k+1)$ as follows:

\[
(k + 1)^3 + 2(k + 1) = k^3 + 3k^2 + 3k + 1 + 2k + 2 \\
= k^3 + 2k + 3(k^2 + k + 1)
\]

Observe that $3|k^3 + 2k$ by the induction hypothesis and $3|3(k^2 + k + 1)$. So $3|k^3 + 2k + 3(k^2 + k + 1)$.

Therefore by the principle of mathematical induction we conclude that $n^3 + 2n$ is divisible by 3, for all $n \in \mathbb{N}, n \geq 1$. □