CISC-102 FALL 2016

HOMEWORK 8
SOLUTIONS

(1) Let S be a finite subset of the positive integers. What is the smallest value for $|S|$ that guarantees that at least two elements of $x, y \in S$ that have the same remainder when divided by 100. HINT: Use the pigeon hole principle.

We know that there are 100 equivalent residue classes mod 100. We can view these equivalent classes as the pigoen holes. No matter how we select a subset S of the positive integers of size 101, the pigeon hole principle guarantees that at least two elements of S have the same remainder when divided by 100.

(2) What is the number of ways to colour n identical objects with 2 colours, so that each colour is used at least once?

We start by colouring 2 of the objects one of each colour. There is only one way to do this. This satisfies the requirement that each colour is used at least once. Now count the number of colourings with 2 colours of the remaining $n-2$ objects, that is,

$$\frac{(n-2+1)!}{(n-2)!1!} = \binom{n-1}{1} = \binom{n-1}{n-2}$$

(3) What is the number of ways to colour n different objects with 2 colours, so that each colour is used at least once?

There are 2^n ways to colour n objects with 2 colours. This also counts colourings where only a single colour is used. There are exactly two ways in which this can happen. So the number of colourings using two colours of n different objects such that each colour is used at least once is $2^n - 2$.

(4) How many 5 card hands are there (unordered selection from a standard 52 card deck) that consist of a single pair of the same value, and three other cards of different values? Two possible examples are:

$$2\heartsuit, 2\diamondsuit, 7\clubsuit, 9\diamondsuit 3\heartsuit \text{ and } A\heartsuit, A\clubsuit, 4\diamondsuit, 6\diamondsuit 3\heartsuit$$

First consider the pair. There are 13, or $\binom{13}{1}$ possible values for the pair. Within each value there are $\binom{4}{2}$ ways to select the suits of the cards.

The remaining three cards must come from the remaining 12 choices. There are $\binom{12}{3}$ ways of getting those 3 values without regard to the suit. There are 4, or $\binom{4}{1}$ ways to select the suit for each of these three cards.

Putting this all together we get the product:

$$\binom{13}{1}\binom{4}{2}\binom{12}{3}\binom{4}{1}^3.$$

(5) A skip straight is 5 cards that are in consecutive order, skipping every second rank (for example 3-5-7-9-J). How many 5 card hands are there (unordered selection from a standard 52 card deck) that form a skip straight?

Observe that once the lowest value of the skip straight is selected the values of the 4 remaining cards is pre-determined. The highest the lowest value of a skip straight can be is 6, because 6-8-10-Q-A is a valid skip straight, but if we start at 7 or higher we run out of values at the high end. This leads us to conclude that there are 6, or $\binom{6}{1}$ ways to get a skip straight without concern for the suit of each card. There are 4, or $\binom{4}{1}$ ways to select the suit for each card in the skip straight.

Putting this all together we get the product:

$$\binom{6}{1}\binom{4}{1}^5.$$

Note: This also counts the case where all 5 cards are of the same suit. We can call this a *skip straight flush*. The number of straight skip flushes is:

$$\binom{6}{1}\binom{4}{1}.$$

(6) Use the binomial theorem to expand the product $(x+y)^6$.

$$\binom{6}{0}x^6+\binom{6}{1}x^5y+\binom{6}{2}x^4y^2+\binom{6}{3}x^3y^3+\binom{6}{4}x^2y^4+\binom{6}{5}xy^5+\binom{6}{6}y^6$$

(7) Show that

$$\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\binom{n}{3}+\cdots+\binom{n}{n}=0$$

HINT: Use the Binomial theorem.

Note that this equation can also be written as follows:

$$\sum_{i=0}^{n}\binom{n}{i}(-1^i)=0$$

The binomial theorem with $a=1$ and $b=-1$ can be written as:

$$0=(1-1)^n=\sum_{i=0}^{n}\binom{n}{i}(1^{n-i})(-1^i)$$

And this proves the result.