CISC-102
Fall 2016
Week 2

Idempotent laws:	(1b) $A \cap A = A$
Associative laws:	(2b) $(A \cap B) \cap C = A \cap (B \cap C)$
Commutative laws:	(3b) $A \cap B = B \cap A$
Distributive laws:	(4b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Identity laws:	(5b) $A \cap \mathbf{U} = A$
	(6b) $A \cap \emptyset = \emptyset$

Verify these properties with **U** = {1,2,3,4,5,6,7}, A = {1,2,3}, B = {2, 4, 6} C={4,5}.

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Associative laws:	(2a) $(A \cup B) \cup C = A \cup (B \cup C)$
Commutative laws:	(3a) $A \cup B = B \cup A$
Distributive laws:	(4a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
Identity laws:	(5a) $A \cup \emptyset = A$
	(6a) $A \cup \mathbf{U} = \mathbf{U}$

Verify these properties with U = {1,2,3,4,5,6,7}, A = {1,2,3}, B = {2, 4, 6}.

Involution laws:	(7) $(A^{\mathrm{C}})^{\mathrm{C}} = A$
Complement laws:	(8a) $A \cup A^{\mathrm{C}} = \mathbf{U}$
	(9a) $\mathbf{U}^{\mathrm{C}} = \emptyset$
DeMorgan's laws:	(10a) $(A \cup B)^{\mathrm{C}} = A^{\mathrm{C}} \cap B^{\mathrm{C}}$

Verify these properties with U = {1,2,3,4,5,6,7}, A = {1,2,3}, B = {2, 4, 6}.

Complement laws:	(8b) $A \cap A^{\mathrm{C}} = \emptyset$
	(9b) $\emptyset^{\mathrm{C}} = \mathbf{U}$
DeMorgan's laws:	(10b) $(A \cap B)^{\mathrm{C}} = A^{\mathrm{C}} \cup B^{\mathrm{C}}$

Verify these properties with U = {1,2,3,4,5,6,7}, A = {1,2,3}, B = {2, 4, 6}.

Definition: A set S is said to be ***finite*** if S is empty or if S contains exactly m elements where m is a positive integer; otherwise S is ***infinite***.

Some finite sets:
A = $\{1, 2, 3, 4\}$,
B = $\{x : x \in \mathbb{N}, x \geq 1, x \leq 6\}$1

Some infinite sets:
C = $\{1, 2, 3, 4, \dots \}$,
D = $\{x : x \in \mathbb{R}, x \geq 1, x \leq 6\}$

Notation: We use vertical bars | | to denote the size or cardinality of a finite set.

$| A | = 4$.

B = $\{1,2,3,4,5,6\}$, so $| B | = 6$.

Definition: The set of all (different) subsets of S is the ***power set*** of S, which we denote as P(S).

If S is a finite set we can prove that:

$| P(S) | = 2^{|S|}$.

Some examples:

$\varnothing$: the empty set has 0 elements, and 1 subset. So $| P(\varnothing) | = 2^0$.

$\{a\}$: has 1 element and 2 subsets.
So $|P(\{a\})| = 2^1$.

$\{a,b\}$: has 2 elements and 4 subsets.
So $|P(\{a,b\})| = 2^2$.

{a,b,c}: Let's keep track of the subsets of {a,b,c} by using a binary string counter.

Corrolary: There are 2 types of people in the world, those who can spell "those" and "don't" correctly and those that can't.

```
a b c
1 1 1
```
denotes {a,b,c}

```
a b c
1 0 1
```
denotes {a,c}

```
a b c
0 0 0
```
denotes $\varnothing$

We can keep track of the subsets of {a,b,c} by using a 0 or 1 in a 3 bit binary string to denote the presence or absence of the symbol in the subset.

For elements of {a,b,c} we say that
a is element 1, b is element 2, c is element 3

We also say that the binary bits are numbered 1,2,3 from left to right.

To map a subset to a binary number
set bit i to
0 if the element i **is not in** the subset
1 if the element i **is in** the subset

Claim: Every different 3 bit binary string denotes a different subset of {a,b,c}, and every subset of {a, b, c} is represented uniquely by a 3 bit binary string.

If we can count 3 bit binary strings then we can also count subsets of {a,b,c}.

There are 2 choices for the left bit (bit 1).
There are 2 choices for the middle bit (bit 2).
There are 2 choices for the right bit (bit 3).

The total number of choices are:

$$2 \times 2 \times 2 = 2^3 = 8.$$

This coincides with the claim that:

$$|P(\{a,b,c\})| = 2^3 .$$

A standard "trick" in mathematics is to obtain a mapping from a new problem to one where the solution is known. When done properly (the mapping has to be one-to-one and onto) the known solution can be used to solve a new problem.

Definition: Let S be a non-empty set. A *partition* of S consists of disjoint non-empty subsets of S whose union is S.

For example: Let S be the set $\{1,2,3,4,5,6\}$:
$E = \{x \in S : x \text{ is even}\}$
$O = \{x \in S : x \text{ is odd}\}$

Observe that $E \cap O = \varnothing$ and $E \cup O = S$, so the set (of sets) $\{E,O\}$ is a partition of S.

This $\{\{1,2\},\{4,3,6\},\{5\}\}$ is another partition of S.

When solving algorithmic problems it is sometimes useful to partition the problem into cases.

A collection or a class of sets can be defined using an index as follows:

Let A_i denote the set $\{x : x \in \mathbb{Z}, x \geq i\}$, for all $i \in \mathbb{N}$. So we have:

$A_1 = \{ 1, 2, 3, ... \}$, $A_2 = \{ 2, 3, 4, ... \}$, $A_3 = \{ 3, 4, 5, ... \}$ etc.

A shorthand notation can be used to denote multiple unions and intersection operations.

For example:

$$A_1 \cup A_2 \cup A_3$$

can be written as:

$$\cup_{i=1}^{3} A_i$$

This is a convenient way to define a large class of sets (in this case infinitely many) and leads to some interesting consequences.

1. $A_i \subseteq A_j$ if $i \geq j$

2. $\bigcup_{i \in \mathbb{N}} A_i = \mathbb{N}$

3. $\bigcap_{i \in \mathbb{N}} A_i = \emptyset$

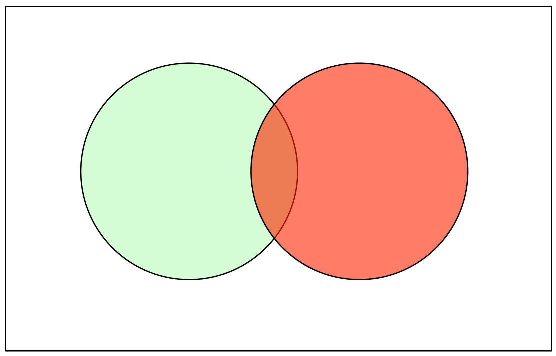
To justify 3. suppose that there is at least one element k in the "intersection". So k must be a positive integer. However by definition there is a set A_ℓ such that $\ell > k$ and $k \notin A_\ell$. This contradicts the assumption that there is at least one element in the "intersection" and therefore we conclude that the "intersection" is equal to $\emptyset$.

Principle of Inclusion and Exclusion

Ginger owns a peculiar music store where all of the instruments are either red or if the instrument is not red it is a guitar.

There are 15 red instruments and 18 guitars. How many instruments are there at Ginger's?

Here's a Venn diagram modelling the inventory at Ginger's.



The diagram clearly shows that we also need to know how many of the guitars at Ginger's are red to determine the total number of instruments in the store. So suppose there are 8 red guitars at Ginger's.

Let R denote the set of red instruments at Ginger's and G the set of guitars. We then have:

$|R| = 15$, $|G| = 18$, $|R \cap G| = 8$.

The total number of instruments is:

$|R \cup G| = |R| + |G| - |R \cap G| = 15 + 18 - 8 = 25$.

The Principle of Inclusion and Exclusion can be stated as follows:

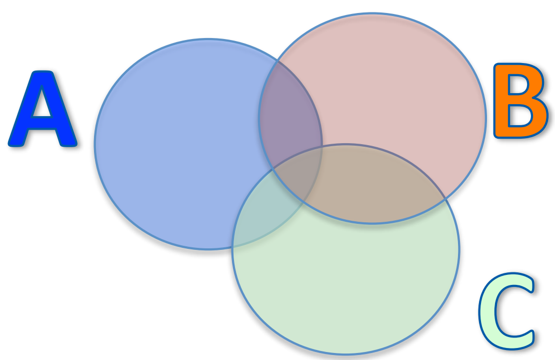
Theorem: Suppose A and B are finite sets. Then:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

This generalizes to a formula for determining the cardinality of the union of three sets.

Corollary: Suppose A, B, and C are finite sets. Then:

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$



Consider a collection of 40 people where each of them is wearing something that is red or blue or green such that:

20 wear something blue,
20 wear something red,
20 wear something green
10 wear red and blue, 10 wear red and green, 10 wear blue and green.

How many people in the class are wearing all 3 colours?

We will come back to the Principle of inclusion and exclusion when we look at more counting problems.