CISC-102
Fall 2016
Week 4

Functions

We have already seen functions in this course. For example:

$$x^2 - 4x + 3$$

We could also write this function as an equation:

$$y = x^2 - 4x + 3$$

In this example you can think of plugging in a (Real) value for x and you will get a distinct value for y. So functions can be viewed as a *mapping* or a *transformation* or even some kind of *machine or algorithm* that takes an input an produces a distinct output.

Underlying every function are two sets (the two sets can be the same).

Let A and B these two sets. We define a function f from A into B as a mapping from every element of A to a distinct element of B. This can be written as:

$$f : A \rightarrow B$$

Vocabulary

Suppose f is a function from the set A to the set B. Then we say that A is the ***domain*** of f and B is the ***codomain*** of f. (Synonyms for codomain are: *target set* & *range*)

Notation

Let f denote a function from A to B, then we write:

$$f : \mathrm{A} \rightarrow \mathrm{B}$$

which is pronounced "f is a function from A to B", or "f maps A into B".

If $a \in$ A, and $b \in$ B we can write:

$$f(a) = b$$

to denote that the function f maps the element a to b.

More Vocabulary

We can say that b is the ***image*** of a under f.

More notation.

A function can be expressed by a formula (written as an equation, as illustrated by the following example:

$$f(x) = x^2 \text{ for } x \in \mathbb{R}$$

In this example f is the function and x is the variable.

Sometimes we can express the image of a variable (the *independent variable*) by a *dependent variable* as follows:

$$y = x^2$$

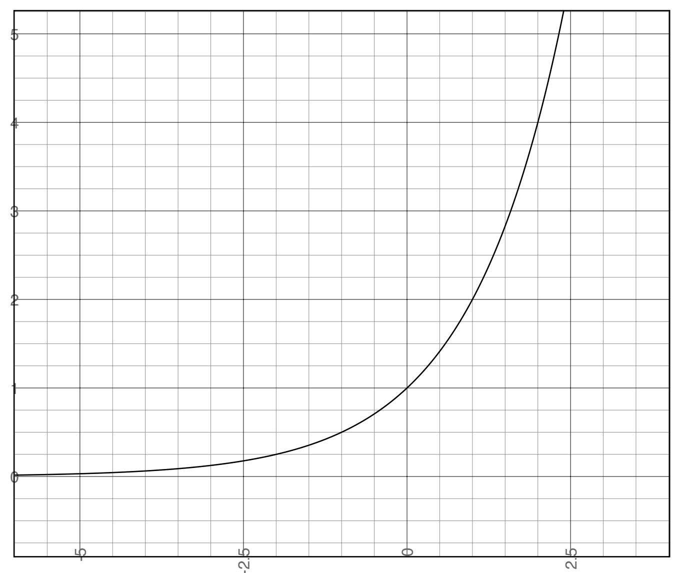
Injective(one-to-one), Surjective(onto), Bijective(one-to-one and onto) functions.

A function f: A → B is a *one-to-one* function if for every

a ∈ A there is a distinct image in B. A one-to-one function is also called an *injective function* or an *injection*.

Let $f : \mathbb{R} \to \mathbb{R}$ and $f(x) = 2^x$.

$f(x) = 2^x$ is one-to-one because there is a distinct image for every $x \in \mathbb{R}$, that is if $2^x = 2^y$ then $x = y$.

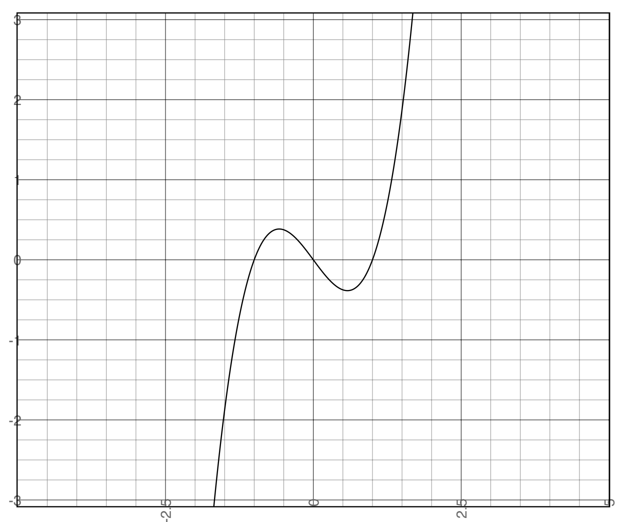


A function f: A → B is an *onto* function if

every $b \in$ B is an image. An onto function is also called a *surjective function* or a *surjection*.

Let $f : \mathbb{R} \to \mathbb{R}$ and $f(x) = x^3 - x$.

$f(x) = x^3 - x$ is onto because the pre-image of any real number y is the solution set of the cubic polynomial equation $x^3 - x - y = 0$ and every cubic polynomial with real coefficients has at least one real root.



Note: $f(x) = x^3 - x = x(x^2 - 1)$ is **not** one-to-one because $f(x) = 0$ for $x = -1, x = +1, x = 0$

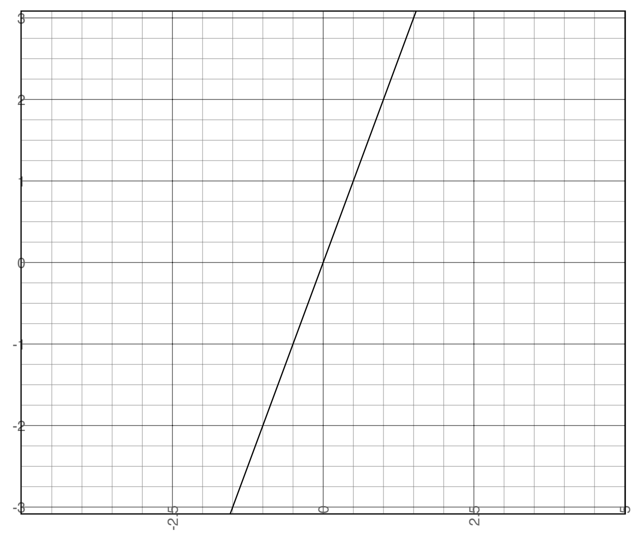
Note: $f(x) = 2^x$ is **not** onto because $2^x > 0$ for all $x \in \mathbb{R}$.

A function that is both one-to-one and onto is called a *bijective function* or a *bijection*.

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ and $f(x) = 2x$

$f(x) = 2x$ is one-to-one because we get a distinct image for every pre-image.

$f(x) = 2x$ is onto because every $y \in \mathbb{R}$ is an image. So $f(x) = 2x$ is a bijection.

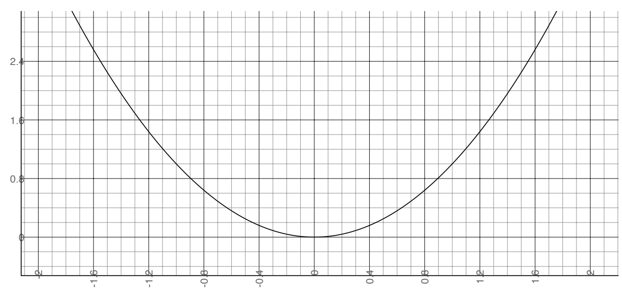


Bijective functions are also called *invertible* functions. That is suppose that f is a bijective function on the set A. Then f^{-1} denotes the inverse of the function f , meaning that whenever $f(x) = y$ we have $f^{-1}(y) = x$.

In our previous example we saw that function $f(x) = 2x$ is a bijective function. In this case we can define

$f^{-1}(x) = x/2$, so we get $f^{-1}(2x) = x$.

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ and $f(x) = x^2$



Observe that $f(x) = x^2$ is a function because every $x \in \mathbb{R}$ has a distinct image. However, $f(x) = x^2$ is neither one-to-one (because $f(x) = f(-x)$) or onto ($f(x) \geq 0$).

Composition of functions

Notation: Suppose we have functions $f : A \to B$

and $g : B \to C$, then the composition of f and g written as

$g \circ f$ is defined as:

$(g \circ f)(a) = g(f(a))$. (NOTE: carefully notice the order of f and g on the two sides of the equation.)

So for example let $f: \mathbb{R} \to \mathbb{R}$ *be* $f(x) = x^2$ and let $g: \mathbb{R} \to \mathbb{R}$ be $g(x) = 5x$. Then an example of a composition of f and g could be:

$(g \circ f)(2) = g(f(2)) = g(4)= 20.$

Recursively Defined Functions

Recall the factorial function, $n!$. We can define n! and (n+1)! using these explicit iterative formulae:

$$n! \quad = \quad 1 \times 2 \times 3 \times \cdots \times n$$
$$(n+1)! \quad = \quad 1 \times 2 \times 3 \times \cdots \times n \times (n+1)$$

Notice how $(n+1)! = n! \times (n+1)$. This is a recursive definition of the factorial function. More formally we have the following definition.

The Factorial function is defined for non-negative integers, that is {0, 1, 2, 3, …} as follows:

(i) If $n = 0$ then $n! = 1$ (Base)
(ii) If $n > 0$ then $n! = n \times (n-1)!$ (Recursive definition)

Definition: (from SN) A function is said to be recursively defined if it has the following two properties:

i) There must be base values that are given and where the function does not refer to itself.
ii) Each time the function does refer to itself the referred function argument must be closer to the base that the referring function argument.

(In the factorial definition (n-1) is closer to 0, than n is.

We can use a recursive definition for the handshake problem.

Suppose that S is a set consisting of *n* elements, $n \geq 2$.
Q. How many two element subsets are there of the set S?

We need to come up with a base statement and a recursive definition.

The recursive definition is based on the observation, a set of *n* elements has *n*-1 more two element subsets than a set of *n-1* elements.

Let f be a function with domain {2,3,4, …} and range $\mathbb{N}$, such that:

i) $f(2) = 1$ (1 two element subset}
ii) $f(n) = f(n-1) + n-1$.

We can use a recursive definition for the number of values that can be stored in a binary string. The recursive definition is based on the observation that an n bit binary number stores twice as many bits as an (n-1) bit binary number.

Let f be a function on the the Natural numbers such that:

i) $f(1) = 2$ (2 values can be stored in one bit)
ii) $f(n) = f(n-1) \times 2$

Relations (See chapter 2. of SN)

An ordered pair of elements a,b is written as (a,b).

NOTE: Mathematical convention distinguishes between

"( )" brackets -order is important – and "{ }" -- not ordered.

Example: $\{1,2\} = \{2,1\}$, but $(1,2) \neq (2,1)$.

Product sets

Let A and B be two arbitrary sets. The set of all ordered pairs (a,b) where $a \in A$ and $b \in B$ is called the *product* or *Cartesian product* [1] or *cross product* of A and B.

The cross product is denoted as:

$$A \times B = \{(a,b) : a \in A \text{ and } b \in B\}$$

and is pronounced "A cross B". It is common to denote $A \times A$ as A^2.

A "famous" example of a product set is $\mathbb{R}^2$, that is the product of the Reals, or the two dimensional real plane or Cartesian plane -- x and y coordinates.

[1] Réne Descartes French philosopher mathematician (1596 - 1650)

Relations

Definition: Let A and B be arbitrary sets. A ***binary relation***, or simply a ***relation*** from A to B is a subset of $A \times B$.

(We study relations to continue our exploration of mathematical definitions and notation.)

Example: Suppose $A = \{1,3,6\}$ and $B = \{1,4,6\}$

$A \times B = \{(a,b) : a \in A, \text{ and } b \in B \}$

$= \{(1,1),(1,4),(1,6),(3,1),(3,4),(3,6),(6,1),(6,4)(6,6)\}$

Example: Consider the relation $\leq$ on $A \times B$ where A and B are defined above. The subset of $A \times B$ in this relation are the pairs:

$\{(1,1),(1,4),(1,6),(3,4),(3,6),(6,6)\}$

That is, a pair (a,b) is in the relation $\leq$ whenever $a \leq b$.

<u>Vocabulary</u>
When we have a relation on $S \times S$ (which is a very common occurrence) we simply call it a relation *<u>on</u>* S, rather than a relation on $S \times S$.

Let $A = \{1,2,3,4\}$, we can define the following relations on A.

$R_1 = \{(1,1), (1,2), (2,3), (1,3), (4,4)\}$

$R_2 = \{(1,1), (1,2), (2,1), (2,2), (3,3), (4,4)\}$

$R_3 = \{(1,3), (2,1)\}$

$R_4 = \varnothing$

$R_5 = A \times A = A^2$ (How many elements are there in R_5 ?)

Properties of relations on a set A

Reflexive: A relation R is reflexive if $(a,a) \in R$ for all $a \in A$.

Symmetric: A relation R is symmetric if **whenever** $(a_1, a_2) \in R$ then $(a_2, a_1) \in R$.

Antisymmetric: A relation R is antisymmetric if **whenever** $(a_1, a_2) \in R$, and $a_1 \neq a_2$, then $(a_2, a_1) \notin R$.

An alternate way to define antisymmetric relations (as found in Schaum's Notes) is:

Antisymmetric: A relation R is antisymmetric if **whenever** $(a_1, a_2) \in R$ and $(a_2, a_1) \in R$ then $a_1 = a_2$.

NOTE: There are relations that are neither symmetric nor antisymmetric or both symmetric and antisymmetric.

Transitive: A relation **R** is transitive if **whenever** $(a_1, a_2) \in R$ and $(a_2, a_3) \in R$ then $(a_1, a_3) \in R$.

Let $A = \{1,2,3,4\}$, we can define the following relations on A.

$R_1 = \{(1,1), (1,2), (2,3), (1,3), (4,4)\}$

NOT reflexive, NOT symmetric, antisymmetric, transitive

$R_2 = \{(1,1), (1,2), (2,1), (2,2), (3,3), (4,4)\}$

reflexive, symmetric, NOT antisymmetric, transitive

$R_3 = \{(1,3), (2,1)\}$

NOT reflexive, NOT symmetric, antisymmetric, NOT transitive

$R_4 = \varnothing$

NOT reflexive, symmetric, antisymmetric, transitive

$R_5 = A \times A = A^2$ (How many elements are there in R_5 ?)

reflexive, symmetric, transitive.

Consider the relation

$R_6 = \{(1,1), (1,2), (2,1), (2,3),(2,2), (3,3)\}$

NOT reflexive, NOT symmetric, NOT antisymmetric, NOT transitive

Consider the relations <, ≤, and = on the Natural numbers. (less than, less than or equal to, equal to)

The relation < on the Natural numbers $\{(a,b) : a,b \in N, a < b\}$ is:

NOT reflexive, NOT symmetric, antisymmetric, transitive

The relation ≤ is on the Natural numbers $\{(a,b) : a,b : \in N, a < b\}$ is:

reflexive, NOT symmetric, antisymmetric, transitive

The relation = on the Natural numbers $\{(a,b) : a,b : \in N, a = b\}$ is:

reflexive, symmetric, NOT antisymmetric, transitive

Partial orders and equivalence relations

A relation R is called a ***partial order*** if R is reflexive, antisymmetric, and transitive.

A relation R is called an ***equivalence relation*** if R is reflexive, symmetric, and transitive.

Functions as relations

A function can be viewed as a special case of relations.

A relation from A to B is a ***function*** if every element $a \in A$ is assigned a unique element of B.