CISC-102
Fall 2016
Week 8

The Pigeon Hole Principle

If there are *n* pigeons, that all must sleep in a pigeon hole, and n-1 pigeon holes, then there is at least one pigeon hole where (at least) 2 pigeons sleep.
This should be obvious! Mathematicians give it a name because it is a useful counting tool.

Can we find two people living in the G.T.A. that have exactly the same number of strands of hair on their heads?

The answer is YES! And we can prove it using the pigeon hole principle.

The population of the G.T.A is more than 6 million. Science tells us that nobody has more that 500,000 strands of hair on their heads.

To solve the problem using the pigeon hole principle we imagine 500,000 pigeon holes labelled from 1, ..., 500,000 and then imagine each resident of the G.T.A. entering the pigeon hole labelled with the number of strands of hair on their head. Since 6 million is greater than 500,000 we deduce that there will be at least one pigeon hole where two or more people have entered.

Can we find 13 people living in the G.T.A. that have exactly the same number of strands of hair on their heads?

Again the answer is yes! Can you argue why?

Can we find 2 pairs of people living in the G.T.A. that have exactly the same number of strands of hair on their heads?

The pigeon hole principle is useless for solving this problem and we leave this as an unsolved mystery.

Let's look at two more applications of the pigeon hole principle.

Find the minimum number n of integers to be selected from S = {1, 2, …, 9} so that the sum of two of the integers is guaranteed to be even.

If a number x is odd then $x = 2p + 1$ for some integer p. And similarly an odd number y yields, $y = 2q + 1$ for some integer q. Thus $x + y = 2(p+q + 1)$ and is divisible by two. Similarly one can show that the sum of 2 even numbers is even.

This leads to the observation that as long as we have two odd or two even integers we get an even sum, so we partition S into even and odd numbers. By the pigeon hole principle 3 numbers from S will always contain a pair that sums to an even number.

Pigeon holes are: {1,3,5,7,9} and {2,4,6,8}

Find the minimum number n of integers to be selected from S = {1, 2, …, 9} so that the absolute difference between two of the integers is exactly 5.

We partition S into pairs that yield a difference of 5.

Pigeon holes are: {1,6},{2,7},{3,8},{4,9},{5}

So we need to pick 6 numbers to guarantee that difference of two is 5.

Counting

Suppose that we have n identical objects and 3 cans of paint one red, one blue, and one green. We can assume that there is enough paint in each can to colour all of the objects.

How many different ways are there to colour the objects so that each object gets only one colour?

This counting problem uses the paradigm of selecting balls from a bag without regard to ordering and replacing each ball back into the bag after it has been selected. This can be abbreviated as *selection with replacement and without ordering*.

We can model this as counting binary strings using n 0's and 2 1's. There are:

(n+2)! / (2! n!) ways to do this. (There are n+2 symbols where 2 repeat (the 1's) and n repeat (the 0's).

Note that (n+2)! / (2! n!) can also be written as the binomial coefficient

$$\binom{n+2}{2}$$

We can think of this as a string of length n+2 of all 0's, and we select two (different) 0's to convert into 1's.

Suppose that we insist that each colour is used at least once. How many ways are there to colour n identical objects with 3 colours so that each colour is used at least once.

We can think of this as pre-assigning one of the objects to each colour. So now we count the number of binary strings of length n - 3 + 2 = n-1 consisting of n-3 0's, and 2 1's.

Suppose that we have n different objects and 3 cans of paint one red, one blue, and one green. We can assume that there is enough paint in each can to colour of all of the objects.

How many different ways are there to colour the objects so that each object gets only one colour?

This counting problem uses the paradigm of selecting balls from a bag with regard to ordering and replacing each ball back into the bag after it has been selected. This can be abbreviated as *selection with replacement and with ordering*.

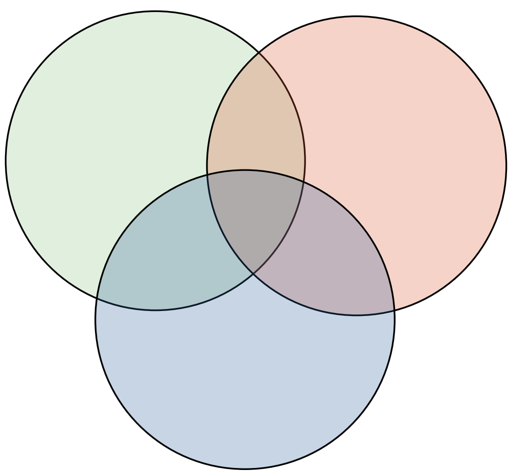
Thus we get 3^n ways to paint the objects with three colours.

Suppose that we insist that each colour is used at least once. How many ways are there to colour n different objects with 3 colours so that each colour is used at least once.

Recall that the principle of inclusion and exclusion says that the cardinality of the union of 3 sets, R, G, B can be determined as follows:

$|R \cup G \cup B| = |R| + |G| + |B| - |R \cap G| - |G \cap B| - |R \cap B| + |R \cap G \cap B|$

Consider the following Venn diagram:



The green circle represents all colourings that has at least one object coloured green, and |G| represents the number of colourings that have at least one object coloured green.

That is too hard for me to count. I would rather do something easier.

Consider the parts where only two colours are used. For example colouring the objects with only red and green. In notation we would write that as

$(R \cap G) \setminus B$

with cardinality

$| (R \cap G) \setminus B | = 2^n$.

Note that this also counts the number of ways to colour the objects where only a single colour (red, or green) is used.

Again we can immediately deduce that:

$| (B \cap G) \setminus R | = | (R \cap B) \setminus G | = 2^n$.

The part of the green circle that is not in the intersection of either of the circles, represents the number of ways of colouring the objects so that they are all green.

Using our set theory notation this is,

$G \setminus (R \cup B)$

and the cardinality of this set is:

$| G \setminus (R \cup B) | = 1.$

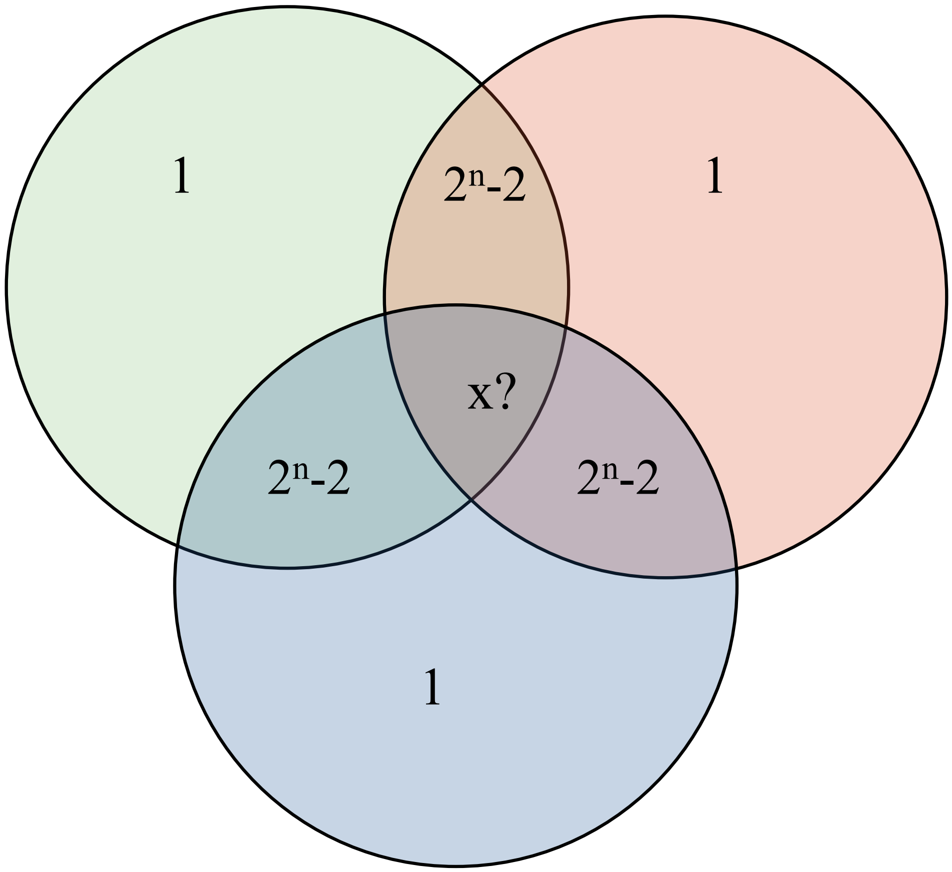
That is there is only one way to colour all objects green.

Using the same type of reasoning we get:

$| R \setminus (G \cup B) | = | B \setminus (R \cup G) | = 1.$

We already know that

$|R \cup G \cup B| = 3^n.$



Given this diagram we can solve for x.

$3^n = 3(2^n - 2) + 3 + x$, so we get:

$x = 3^n - 3(2^n) + 3$.

So we have $3^n - 3(2^n) + 3$ ways to colour n different objects using at least one of each of the three colours.

Counting Poker Hands.

Notation:

A card from a standard 52 card deck will be denoted using an ordered pair as follows:

(v,s) where v is an element of the set of 13 values:

$$\{A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, K, Q\}.$$

and s is an element of the set of 4 suits>

$$\{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}.$$

is the suit of the card.

For this discussion a poker hand is a 5 card subset of the 52 card deck.

There are 2,598,960 different 5 card subsets from a 52 card deck.

How is this value obtained?

The most valuable poker hand is a royal flush, that is a 5 card subset that consists of the values 10,J,Q,K,A all in the same suit.

For example:

{(10,♧), (J,♧), (Q,♧), (K, ♧), (A,♧)}

is one example of a royal flush.

How many royal flushes are there?

The odds of getting a royal flush is: 649,739 : 1.
How do we obtain this value?

The next highest hand is a straight flush. That is a hand of 5 consecutive values (where A = 1 or A =14 as appropriate) all of the same suit. Normally the designation straight flush excludes the royal flushes.

There are a total of

$$\binom{10}{1}\binom{4}{1}-\binom{4}{1}$$

straight flushes.

A four of a kind consists of 4 cards of the same value plus one additional card.

Let's look at two equivalent ways of counting the number of 4 of kind hands.

1. Count the number of ways to select the value of the four of a kind (13) and then the number of ways to choose the 5th card (48), and multiply.
2. Count the number of ways to select the value of the four of a kind (13) and then number of ways to select the suit of the value of the 5th card (12) and the number of ways to select the suit of the 5th card (4), and multiply.

The odds of getting a four of a kind is 4,164 : 1

To see why we compute the product:

13(48) = 624

So there are 2,598,960 - 624 = 2,598,336 ways to get a "non four of a kind" vs. 624 ways to get a 4 of a kind, giving:

2,598,336:624 odds

which simplifies to:

4,164 : 1

A full house consists of 3 cards of the same value plus 2 cards of the same value?

For example: $\{7\clubsuit, 7\diamondsuit, 7\spadesuit, 3\heartsuit, 3\spadesuit\}$ is a full house.

There are:

$$\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}$$

ways to get a full house.

A poker hand is called 3 of a kind, when 3 cards have the same value, and the other two can be any two of the remaining values.

For example: $\{7\clubsuit, 7\diamondsuit, 7\spadesuit, 2\heartsuit, 3\spadesuit\}$ makes 3 of a kind.

How many different 3 of a kind hands are there?

A poker hand is called two pair if it consists of two distinct pairs of the same value and a 5th card with value different from the first two.

An <u>incorrect</u> way to count this is:

There are

$$\binom{13}{1}\binom{4}{2}$$

ways to get the first pair and

$$\binom{12}{1}\binom{4}{2}$$

ways to get the second pair and

$$\binom{11}{1}\binom{4}{1}$$

to get the 5th card.
Putting this together we get

$$\binom{13}{1}\binom{12}{1}\binom{4}{2}^2\binom{11}{1}\binom{4}{1}$$

Can you detect the error?

1st pair is {2♢,2♧}, 2nd pair is {3♡,3♢} and the 5th card is {J♧}.

Observe that this is the same hand as:

1st pair is {3♡,3♢}, 2nd pair is {2♢,2♧}and the 5th card is {J♧}.

So we count each hand twice the correct expression is:

$$\binom{13}{2}\binom{4}{2}^2\binom{11}{1}\binom{4}{1}$$

A poker hand is called a straight if it consists of 5 values in a row. For the purposes of this question we will exclude hands that are straight flushes.

We already know that the number of straight flushes (including royal flushes) is

$$\binom{10}{1}\binom{4}{1}$$

This assumes that all cards are of the same suit. In a straight the suit of each of the 5 cards is open to selection.

So the number of straights (including straight flushes) is:

$$\binom{10}{1}\binom{4}{1}^5$$

The final step to get the correct count is to subtract the straight flushes.

$$\binom{10}{1}\binom{4}{1}^5 - \binom{10}{1}\binom{4}{1}$$

A hand with no straight or flush or 4,3, or 2 of a kind is called a no-pair. How do we count the number of 5 card no-pair hands.

A counting idea we can exploit is:

- Count the number of ways to get 5 different cards that are not a straight.
- Count the different suits for these 5 cards that do not make a flush.

So the number of 5 card no-pair hands is:

$$\left(\binom{13}{5} - \binom{10}{1}\right)\left(\binom{4}{1}^5 - \binom{4}{1}\right)$$

The probability of getting a no-pair when randomly selecting 5 cards is:

$$\frac{\left(\binom{13}{5} - \binom{10}{1}\right)\left(\binom{4}{1}^5 - \binom{4}{1}\right)}{\binom{52}{5}}$$

And this works out to be:

(1277)(1020)/2,598,960[1] = 1,302,540/2,598,960
or about 0.5.

You can find counting formulas for a variety of 5 card poker hands on this Wikipedia page:

https://en.wikipedia.org/wiki/Poker_probability

[1] I used Octave, an open source version of Matlab, to compute this value.

The Binomial Theorem

When we expand the expression:
$(x + y)^3$

we get:
$(x+y)(x+y)(x+y) = x^3 + 3x^2y + 3xy^2 + y^3$

this can also be written as follows:

$$(x+y)(x+y)(x+y) = \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3$$

We can reason that when we expand *$(x + y)^3$,* there is one way to choose a triple that is exclusively x's (with 0 y's), 3 ways to choose a triple that has 2 x's (and 1 y) , and 3 ways to choose a triple that has 1 x (and 2 y's). Finally there is 1 way to choose a triple with no x (and 3 y's).

Binomial Theorem:

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n}x^0 y^n$$
$$= \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k$$

For all natural numbers *n*.

Proof: In the expansion of the product:

$$(x + y)\,(x + y) \cdots (x+y),$$

there $\binom{n}{k}$ ways to choose an *n*-tuple with n-*k* *x*'s and (*k* *y*'s). $\square$

A special case of the binomial theorem should look familiar.

$$
\begin{aligned}
(1+1)^n &= \binom{n}{0}1^n1^0 + \binom{n}{1}1^{n-1}1 + \binom{n}{2}1^{n-2}1^2 + \cdots + \binom{n}{n}1^0 1^n \\
&= \sum_{k=0}^{n}\binom{n}{k}
\end{aligned}
$$

This is just the sum the sizes of all subsets of a set of size *n*.