CISC-102
Fall 2016
Week 9

## Using counting to prove theorems.

Counting arguments can be useful tool for proving theorems. In each case there is also an algebraic of proving the result. However, there is an inherent beauty in the elegant simplicity of some of these counting arguments so it's well worth looking at some examples. These proofs lack the formality of algebraic proofs. The lack of formality may make these arguments harder to grasp for some, and easier to understand for others.

The proofs we see will be to prove the validity of equations. We will count the left and right hand side of each equation and show that they count the same thing.

## Binomial Coefficients

We prove identities involving binomial coefficients using counting arguments.

## Theorem:

$$\binom{n}{k} = \binom{n}{n-k}$$

**Proof:** On the left we have the quantity $\binom{n}{k}$ which represents the number of ways to select a $k$ element subset from an $n$ element set, $S$. Using the analogy of selecting balls from a bag, we see that we also implicitly select the complementary subset that stays in the bag, and the number of ways to do this is as given on the right hand side of the equation is $\binom{n}{n-k}$. $\square$

## Theorem A:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

**Proof:** On the left the quantity $\binom{n}{k}$ represents the number of ways to select a $k$ element subset from an $n$ element set, $S$. To see what the right hand side counts we suppose that there is a "favourite" or "distinguished" element of the set, call it $x$. The number of ways to select a $k$ element subset from $n$ distinct objects that is guaranteed to <u>*include*</u> $x$ is to pull $x$ out and then choose the remaining *n-1* elements in $\binom{n-1}{k-1}$ ways. On the other hand the number of ways to select a $k$ element subset from $n$ distinct objects that is guaranteed to <u>*exclude*</u> $x$ is to pull $x$ out and then choose all $k$ elements in $\binom{n-1}{k}$.

Therefore the left and right hand side both count the same thing thus justifying the equation. $\square$

And here's an alternate proof for Theorem A.
**Theorem A:**

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$$

**Proof:**

$$
\begin{aligned}
\binom{n}{k-1} + \binom{n}{k} &= \frac{n!}{(n-k+1)!(k-1)!} + \frac{n!}{(n-k)!(k)!} \\
&= \frac{n!k + n!(n-k+1)}{(n+1-k)!k!} \\
&= \frac{n!(k + n - k + 1)}{(n+1-k)!k!} \\
&= \frac{n!(n+1)}{(n+1-k)!k!} \\
&= \frac{(n+1)!}{(n+1-k)!k!} \\
&= \binom{n+1}{k}
\end{aligned}
$$

## Theorem:

$$\sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$$

**Proof:** On the left the sum counts all the subsets of a set of size $n$. We already know that the number of subsets of a set of size $n$, is $2^n$.

Therefore the left and right hand side both count the same thing thus justifying the equation. $\square$

## The Binomial Theorem

When we expand the expression:

$$(x + y)^3$$

we get:

$$(x+y)(x+y)(x+y) = x^3 + 3x^2y + 3xy^2 + y^3$$

this can also be written as follows:

$$(x + y)(x + y)(x + y) = \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3$$

We can reason that when we expand $(x + y)^3$, there is one way to choose a triple that is exclusively x's (with 0 y's), 3 ways to choose a triple that has 2 x's (and 1 y) , and 3 ways to choose a triple that has 1 x (and 2 y's). Finally there is 1 way to choose a triple with no x (and 3 y's).

## Binomial Theorem:

$$(x + y)^n = \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \cdots + \binom{n}{n} x^0 y^n$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

For all natural numbers $n$.

**Proof:** In the expansion of the product:

$$(x + y)(x + y) \cdots (x+y),$$

there $\binom{n}{k}$ ways to choose an $n$-tuple with n-$k$ $x$'s and ($k$ $y$'s). $\square$

We can use Theorem A to prove the binomial theorem using mathematical induction.

**Binomial Theorem:** Let P(n) be the proposition

$$(x + y)^n = \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \cdots + \binom{n}{n} x^0 y^n$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

for all natural numbers $n$.

**Proof:** (By induction on n)

**Base:** n = 1, (x+y) = 1 × x + 1 × y
**Induction Hypothesis:** Let m be an integer, m ≥ 1. We assume that:

$$(x + y)^m = \sum_{k=0}^{m} \binom{m}{k} x^{m-k} y^k$$

# Induction step:

$$(x+y)^{m+1} = (x+y)(x+y)^m$$

$$= (x+y)\sum_{k=0}^{m}\binom{m}{k}x^{m-k}y^k$$

$$= x\binom{m}{0}x^m +$$

$$y\binom{m}{0}x^m + x\binom{m}{1}x^{m-1}y +$$

$$\cdots +$$

$$y\binom{m}{j-1}x^{m-j+1}y^{j-1} + x\binom{m}{j}x^{m-j}y^j +$$

$$\cdots +$$

$$y\binom{m}{m-1}x^{m-m+1}y^{m-1} + x\binom{m}{m}x^{m-m}y^m +$$

$$y\binom{m}{m}y^m$$

## Proof (continued)

$$= \binom{m}{0} x^{m+1} +$$

$$= \binom{m+1}{1} x^m y$$

$$\cdots +$$

$$\binom{m+1}{j} x^{m+1-j} y^j +$$

$$\cdots +$$

$$\binom{m+1}{m} xy^m +$$

$$\binom{m}{m} y^{m+1}$$

$$= \sum_{k=0}^{m+1} \binom{m+1}{k} x^{m+1-k} y^k$$

Therefore by the principle of mathematical induction we have shown that P(n) is true for all natural numbers n. □

A special case of the binomial theorem should look familiar.

$$(1+1)^n = \binom{n}{0}1^n1^0 + \binom{n}{1}1^{n-1}1 + \binom{n}{2}1^{n-2}1^2 + \cdots + \binom{n}{n}1^01^n$$

$$= \sum_{k=0}^{n} \binom{n}{k}$$

This is just the sum the sizes of all subsets of a set of size $n$.

## Pascal's Triangle

 An easy way to calculate a table of binomial coefficients was recognized centuries ago by mathematicians in India, China, Iran and Europe. In the west the technique is named after the French mathematician
Blaise Pascal (1623-1662). In the example below each row represents the binomial coefficients as used in the binomial theorem.

$$
\begin{array}{ccccccccccccc}
& & & & & & \binom{0}{0} & & & & & & \\
& & & & & \binom{1}{0} & & \binom{1}{1} & & & & & \\
& & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & & \\
& & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & & \\
& & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & & \\
& \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} & \\
\binom{6}{0} & & \binom{6}{1} & & \binom{6}{2} & & \binom{6}{3} & & \binom{6}{4} & & \binom{6}{5} & & \binom{6}{6}
\end{array}
$$

To obtain the entries by hand in a simple way we can use the identity:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

```
                        1
                     1     1
                  1     2     1
               1     3     3     1
            1     4     6     4     1
         1     5    10    10     5     1
      1     6    15    20    15     6     1
   1     7    21    35    35    21     7     1
1     8    28    56    70    56    28     8     1
```