## CISC-102 <br> Winter 2016 <br> Lecture 13

## Techniques of Counting (Chapter 5)

We have already seen and solved several counting problems.
For example:

- How many subsets are there of a set with $n$ elements?
- How many two element subsets are there of a set with $n$ elements.
- How many different ways can the numbers in 6-49 draw be chosen?
- How many ways can people be seated at a table?

Counting problems are useful to determine resources used by an algorithm (e.g. time and space).

## Product Rule Principle

Let $\mathrm{A} \times \mathrm{B}$ denote the cross product of sets A and B .
Then $|\mathrm{A} \times \mathrm{B}|=|\mathrm{A}| \times|\mathrm{B}|^{1}$
For example suppose you have to pick a main course from: Fish, Beef, Chicken, Vegan. We can write this as the set M (Main), as follows

$$
\mathrm{M}=\{\mathrm{F}, \mathrm{~B}, \mathrm{C}, \mathrm{~V}\}
$$

Furthermore there is also choice of a desert from: Apple pie, Lemon meringue pie, Ice cream. This can be represented as the set D.

$$
\mathrm{D}=\{\mathrm{A}, \mathrm{~L}, \mathrm{I}\}
$$

We use the product rule to determine the total number of possible meals, that is:

$$
|\{\mathrm{F}, \mathrm{~B}, \mathrm{C}, \mathrm{~V}\}| \times|\{\mathrm{A}, \mathrm{~L}, \mathrm{I}\}|=(4)(3)=12
$$

[^0]
## The product rule principle can be stated formally as:

Suppose there is an event M that occurs in $m$ ways and an event D that occurs in n ways, and these two events are independent of each other. Then there are $m \times n$ ways for the combination of the two events to occur.

Note that an event can be considered as a set of outcomes, and the combination of events as a cross product of sets.

## Product Rule Principle

The rule generalizes to any number of independent sets (events). For example with 3 sets:
Let $\mathrm{A} \times \mathrm{B} \times \mathrm{C}$ denote the cross product of sets $\mathrm{A}, \mathrm{B}, \& \mathrm{C}$.
Then $|\mathrm{A} \times \mathrm{B} \times \mathrm{C}|=|\mathrm{A}| \times|\mathrm{B}| \times|\mathrm{C}|$
For k sets we have:
$\left|A_{1} \times A_{2} \times \ldots \times A_{k}\right|=\left|A_{1}\right| \cdot\left|A_{2}\right| \cdot \ldots \cdot\left|A_{k}\right|$

For example, DNA is represented using the 4 symbols:
A C G T.

The number of different strings of length 7 using these symbols is:

$$
4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4=4^{7} .
$$

The number of strings of length k using these 4 symbols is:
$4^{k}$

## Sum Rule Principle

Suppose we have the same mains and deserts as before, but must choose a main or a dessert but not both.

Then we have:

$$
|\{\mathrm{F}, \mathrm{~B}, \mathrm{C}, \mathrm{~V}\} \cup\{\mathrm{A}, \mathrm{~L}, \mathrm{I}\}|=4+3=7
$$

choices.

The sum rule principle can be formally stated as:
Suppose event M can occur in mays and a second event D can occur in n ways. The number of ways that M or D can occur is $m+n$.

Again considering an event as a set of outcomes the sum rule principle can be viewed as counting the size of the union of disjoint sets.

Suppose you can take 1 elective from a list of elective courses, where there are 3 courses from the History department, 4 courses from the English department and 2 course from the Psychology department. This can be formalized as the sets:
$H=\left\{h_{1}, h_{2}, h_{3}\right\}, E=\left\{e_{1}, e_{2}, e_{3}, e_{4}\right\}, P=\left\{p_{1}, p_{2}\right\}$
The total number of choices is:
$|H \cup E \cup P|=|H|+|E|+|P|=3+4+2=9$.

## Playing cards.

Some of the following examples make use of the standard 52 deck of playing cards as shown below.

There are 4 suits (clubs, spades, hearts, diamonds) each consisting of 13 values (Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, King) for a total of 52 cards.


## Permutations

A common paradigm for counting is to imagine selecting labeled balls from a bag, so that no two balls are alike.

A permutation of objects is represented by a record of the order in which balls are pulled out of the bag.

Example: How many ways are there to select 5 different coloured balls from a bag?

$$
5 \times 4 \times 3 \times 2 \times 1=5!
$$

We can relate this to the product rule by thinking of the full bag as the set $B_{5}$, the bag with 4 balls as the set $B_{4}$, the bag with 3 balls $B_{3}$, the bag with 2 balls $\mathrm{B}_{2}$, and with 1 ball $B_{1}$. Thus pulling balls from a bag can be viewed as a combination of the events (sets of outcomes) $\mathrm{B}_{1}, \mathrm{~B}_{2}, \mathrm{~B}_{3}$, $B_{4}, B_{5}$. And the number of ways the combination of these events can occur as:

$$
\left|\mathrm{B}_{1}\right| \times\left|\mathrm{B}_{2}\right| \times\left|\mathrm{B}_{3}\right| \times\left|\mathrm{B}_{4}\right| \times\left|\mathrm{B}_{5}\right|=5 \times 4 \times 3 \times 2 \times 1=5!
$$

Example: How many different ways are there to shuffle a deck of cards?

We can number the cards in a deck from 1 to 52 where 1 is the card on top and 52 is the card on the bottom. So shuffling a deck of cards is equivalent to assigning a unique number from $1 \ldots 52$ to each of the cards.

Observe that there is a bijection between the number of ways to draw balls from a bag, and the number of ways to select positions in a shuffled deck of cards. There are 52 positions to select as represented by the the following expression.

$$
52 \times 51 \times 50 \ldots \times 1=52!
$$

A permutation of the elements of a set is in essence assigning an ordering to a set.

## Permutation rule

There are $n$ ! ways to permute $n$ elements.

[^1]
## Permutation of a Subset

Suppose we want to count the number of ways of selecting 2 coloured balls from a total of 5 coloured balls.

$$
5 \times 4=5!/ 3!
$$

Suppose we want to count the number of ways to make an ordered selection of just 5 of the 52 cards.

$$
52 \times 51 \times 50 \times 49 \times 48=52!/ 47!
$$

different ways.

## NOTATION:

$$
\mathrm{P}(\mathrm{n}, \mathrm{k})=\mathrm{n}!/(\mathrm{n}-\mathrm{k})!
$$

represents the number of permutations of $k$ elements chosen from a collection of $n$ elements.

## Combinations

Suppose on the other hand that we want to count the number of different 5 card poker hands. We are interested in the number of ways of selecting 5 from 52 without regard to the way that they are ordered. We can solve this counting problem by answering the following questions.
(1) How many ways are there to shuffle a 5 card deck?

Answer: 5!
(2) How many ways are there to make an ordered selection of 5 of the 52 cards?

Answer: 52!/47!
(3) How do we put these two answers together to count the number of ways to make an un-ordered selection of 5 of the 52 cards?

Answer: We divide the answer to (2) by the answer to (1), yielding: 52!/(47!5!).

## Combinations

We can use the balls in a bag analogy to count combinations. In this case we count the number of different ways to select distinct balls without ordering. The counting technique is a 2 step process.

1. Count the number of ways to select k balls from a bag of $n$ balls with ordering.
2. Divide by the number of ways to order the k selected balls.

The outcome of this process yields the formula:

$$
\frac{n!}{(n-k)!k!}
$$

We have seen this expression before and the accompanying shorthand, that is:

$$
\frac{n!}{(n-k)!k!}=\binom{n}{k}
$$

NOTATION: $\mathrm{C}(n, k)=\mathrm{P}(n, k) / k!=\binom{n}{k}$

## Permutations with Repetition

How many different ways can we order the letters:

## BABY?

You may be temped to say $4!=24$ different ways, (that is select 4 balls labelled B A B Y from a bag) but upon inspection we see that there are only 12 distinguishable ways to order the letters.

This is the list of all 24 permutations that you see come in pairs.

BABY BABY
BAYB BAYB
BBAY BBAY
BBYA BBYA
BYAB BYAB
BYBA BYBA

ABYB ABYB
ABBY ABBY
AYBB AYBB
YBBA YBBA
YBAB YBAB
YABB YABB

The correct way to count this is $4!/ 2$ ! because two of the letters in B A B Y are identical.

How many ways are there to order the letters CCCB?

## BCCC <br> CBCC

## CCBC <br> CCCB

There are $4!/ 3!=4$ ways
How many ways are there to order the letters BBCC?

## BBCC <br> BCBC <br> BCCB

## CBBC

CBCB
CCBB

There are $4!/ 2!2!=6$ ways

Example: How many ways are there to pick ten coloured balls from a bag where each colour appears twice, so that two balls of the same colour are indistinguishable?

$$
\frac{10!}{2!2!2!2!2!}=\frac{10!}{(2!)^{5}}
$$

The counting formula is: The number of permutations of $n$ objects consisting of $n_{1}, n_{2}, n_{3}, \ldots, n_{r}$ that are alike is:

$$
\frac{n!}{n_{1}!n_{2}!\ldots n_{r}!}
$$

Suppose we have a peculiar deck of cards so that suits are omitted (clubs, diamonds, hearts, spades). So we have 4 identical Aces, 4 identical 2's, and so on, up to 4 identical Kings. In how many ways can we shuffle this peculiar deck?

There are 52 ! ways to shuffle 52 distinct cards. However, there are 4 cards of each value so the number of distinguishable ways to shuffle these cards is:

$$
\frac{52!}{(4!)^{13}}
$$


[^0]:    ${ }^{1}$ Recall: vertical bars represent cardinality, or the number of elements in the set.

[^1]:    Larry has 6 distinguishable pairs of socks. Each day Monday to Saturday he wears a different pair of socks. On Sunday he washes the socks (and goes sock-less). In how many different ways can Larry wear a weeks worth of socks?

