CISC-102
Fall 2016
Lecture 14

Counting and the principle of inclusion and exclusion

Suppose that we have n different objects and 3 cans of paint one red, one blue, and one green. We can assume that there is enough paint in each can to colour of all of the objects.

How many different ways are there to colour the objects so that each object gets only one colour?

Since each object can be coloured in one of three ways we have 3^n different ways to colour the objects.

Suppose that we insist that each colour is used at least once. How many ways are there to colour n objects with 3 colours so that each colour is used at least once.

We can apply the principle of inclusion and exclusion to solve this problem as follows:

“Forbidden colourings” are those where one or more colours is not used.

We can enumerate the Forbidden colourings.

Two (or one) colours are used: 3×2^n

One colour used: 3

Since each of the colourings counted with 2 colourings also counts those with one colouring we apply the principle of inclusion and exclusion.

The number of colourings of n distinguishable objects using the colours red, blue, green, such that each colour is used at least once is counted as follows:

$3^n - 3(2^n) + 3$.

Counting Poker Hands.

Notation:

A card from a standard 52 card deck will be denoted using an ordered pair as follows:

(v,s) where v is an element of the set of 13 values:

$$\{A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, K, Q\}.$$

and s is an element of the set of 4 suits>

$$\{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}.$$

is the suit of the card.

For this discussion a poker hand is a 5 card subset of the 52 card deck.

There are 2,598,960 different 5 card subsets from a 52 card deck.

How is this value obtained?

The most valuable poker hand is a royal flush, that is a 5 card subset that consists of the values 10,J,Q,K,A all in the same suit.

For example:

{(10,♧), (J,♧), (Q,♧), (K, ♧), (A,♧)}

is one example of a royal flush.

How many royal flushes are there?

The odds of getting a royal flush is: 649,739 : 1.
How do we obtain this value?

The next highest hand is a straight flush. That is a hand of 5 consecutive values (where A = 1 or A =13 as appropriate) all of the same suit. Normally the designation straight flush excludes the royal flushes.

There are a total of

$$\binom{10}{1}\binom{4}{1}-\binom{4}{1}$$

straight flushes.

A four of a kind consists of 4 cards of the same value plus one additional card.

Let's look at two equivalent ways of counting the number of 4 of kind hands.

1. Count the number of ways to select the value of the four of a kind (13) and then the number of ways to choose the 5th card (48), and multiply.
2. Count the number of ways to select the value of the four of a kind (13) and then number of ways to select the suit of the value of the 5th card (12) and the number of ways to select the suit of the 5th card (4), and multiply.

The odds of getting a four of a kind is 4,164 : 1

To see why we compute the product:

13(48) = 624

So there are 2,598,960 - 624 = 2,598,336 ways to get a "non four of a kind" vs. 624 ways to get a 4 of a kind, giving:

2,598,336 : 624 odds

which simplifies to:

4,164 : 1

A full house consists of 3 cards of the same value plus 2 cards of the same value?

For example: $\{7\clubsuit, 7\diamondsuit, 7\spadesuit, 3\heartsuit, 3\spadesuit\}$ is a full house.

There are:

$$\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}$$

ways to get a full house.

A poker hand is called 3 of a kind, when 3 cards have the same value, and the other two can be any two of the remaining values.

For example: $\{7\clubsuit, 7\diamondsuit, 7\spadesuit, 2\heartsuit, 3\spadesuit\}$ makes 3 of a kind.

How many different 3 of a kind hands are there?

A poker hand is called two pair if it consists of two distinct pairs of the same value and a 5th card with value different from the first two.

An <u>incorrect</u> way to count this is:

There are

$$\binom{13}{1}\binom{4}{2}$$

ways to get the first pair and

$$\binom{12}{1}\binom{4}{2}$$

ways to get the second pair and

$$\binom{11}{1}\binom{4}{3}$$

to get the 5th card.

Putting this together we get

$$\binom{13}{1}\binom{12}{1}\binom{4}{2}^2\binom{11}{1}\binom{4}{3}$$

Can you detect the error?

1st pair is $\{2\diamondsuit, 2\clubsuit\}$, 2nd pair is $\{3\heartsuit, 3\diamondsuit\}$ and the 5th card is $\{J\clubsuit\}$.

Observe that this is the same hand as:

1st pair is $\{3\heartsuit, 3\diamondsuit\}$, 2nd pair is $\{2\diamondsuit, 2\clubsuit\}$and the 5th card is $\{J\clubsuit\}$.

So we count each hand twice the correct expression is:

$$\binom{13}{2}\binom{4}{2}^2\binom{11}{1}\binom{4}{1}$$