CISC-102
Fall 2016
Lecture 15

You get to pick a box of 10 timbits® and choose as many as you like from the choice of

Chocolate, Sugar, Plain, Glazed

In how many different ways can you choose the tidbits?

The way to model this is to consider a bag with balls labelled C,S,P,G and we count the number of ways to select 10 without ordering and with replacement. That is after we select the ball from the bag, we put it back (we replace it). We record the balls that were recorded but we disregard the ordering.

Suppose the 10 choices in order are

C,S,S,S,P,P,P,G,G,G

There are ways $\frac{10!}{3!^3}$ to order these.

On the other hand suppose the choices in order are:

C,C,C,C,C,C,C,C,C,C

There are 10!/10! = 1 way to order this choice.

It appears that the methods that we have so far studied do not solve this counting problem very easily.

Consider the following seemingly unrelated problem, that is, counting the number of binary strings of length 13 consisting of 10 0's and 3 1's.

For example: 0100010001000

We can count the total number of this type of string as

$$13!/(3!10!)$$

Now consider a bijection from binary strings to donut selections.

I claim that there is a bijective mapping from the string

0100010001000 $\leftrightarrow$ C,S,S,S,P,P,P,G,G,G

The mapping works as follows:

The 10 0's represent timbits®, the 1's act as dividers partitioning the zeros into 4 groups.

What does this 0000000000111 binary string represent?

The general counting formula for a problem that can be modelled by selecting k times with replacement and without ordering for a collection of n distinguishable balls is:

$$\frac{(k+n-1)!}{k!(n-1)!}$$

Observe that this quantity can be written as:

$$\binom{k+n-1}{k}$$

Suppose we have a collection of n identical objects and 3 cans of paint one red, one blue, and one green. We can assume that there is enough paint in each can to colour all of the objects.

How many different ways are there to colour the objects so that each object gets only one colour?

Suppose that we insist that each colour is used at least once. How many ways are there to colour n identical objects with 3 colours so that each colour is used at least once.

The Pigeon Hole Principle

If there are *n* pigeons, that all must sleep in a pigeon hole, and n-1 pigeon holes, then there is at least one pigeon hole where 2 pigeons sleep.

This should be obvious! Mathematicians give it a name because it is a useful counting tool.

Can we find two people living in the G.T.A. that have exactly the same number of strands of hair on their heads?

The answer is YES! And we can prove it using the pigeon hole principle.

The population of the G.T.A is more than 6 million. Science tells us that nobody has more that 500,000 strands of hair on their heads.

To solve the problem using the pigeon hole principle we imagine 500,000 pigeon holes labelled from 1, ..., 500,000 and then imagine each resident of the G.T.A. entering the pigeon hole labelled with the number of strands of hair on their head. Since 6 million is greater than 500,000 we deduce that there will be at least one pigeon hole where two or more people have entered.

Can we find 13 people living in the G.T.A. that have exactly the same number of strands of hair on their heads?

Again the answer is yes! Can you argue why?

Can we find 2 pairs of people living in the G.T.A. that have exactly the same number of strands of hair on their heads?

The pigeon hole principle is useless for solving this problem and we leave this as an unsolved mystery.

Let's look at two more applications of the pigeon hole principle.

Find the minimum number n of integers to be selected from S = {1, 2, …, 9} so that the sum of two of the integers is guaranteed to be even.

If a number x is odd then x = 2p + 1 for some integer p. And similarly an odd number y yields, y = 2q + 1 for some integer q. Thus x + y = 2(p+q + 1) and is divisible by two. Similarly one can show that the sum of 2 even numbers is even.

This leads to the observation that as long as we have two odd or two even integers we get an even sum, so we partition S into even and odd numbers. By the pigeon hole principle 3 numbers from S will always contain a pair that sums to an even number.

Pigeon holes are: {1,3,5,7,9} and {2,4,6,8}

Find the minimum number n of integers to be selected from S = {1, 2, …, 9} so that the absolute difference between two of the integers is exactly 5.

We partition S into pairs that yield a difference of 5.

Pigeon holes are: {1,6},{2,7},{3,8},{4,9},{5}

So we need to pick 6 numbers to guarantee that difference of two is 5.