CISC-102
Fall 2016
Lecture 18

Logical Consequence and Arguments

Consider the expression:

p is true and p implies q is true , as a consequence we can deduce that q must be true.

This is a logical argument, and can be written symbolically as,

$p, p \rightarrow q \vdash q$

where: $p, p \rightarrow q$ is called a sequence of *premises,* and q is called the *conclusion*.
The symbol $\vdash$ denotes a logical consequence.

A sequence of premises whose logical consequence leads to a conclusion is called an *argument*.

Valid Argument

We can now formally define what is meant by a valid argument.

> **The argument $P_1, P_2, P_3, \ldots, P_n \vdash Q$ is valid if and only if $P_1 \wedge P_2 \wedge P_3 \wedge \ldots \wedge P_n \rightarrow Q$ is a tautology.**

Example: Consider the argument

$$p \rightarrow q,\ q \rightarrow r,\ \vdash p \rightarrow r$$

We can see if this argument is valid by using truth tables to show that the proposition:

$$(p \rightarrow q) \wedge (q \rightarrow r) \rightarrow (p \rightarrow r)$$

a tautology, that is, the proposition is true for all T/F values of p,q,r.

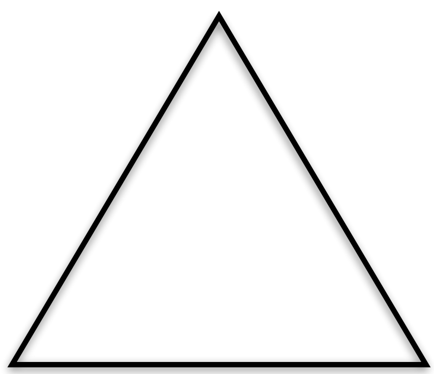
p	q	r	$(p \rightarrow q) \wedge (q \rightarrow r)$	$(p \rightarrow r)$	$(p \rightarrow q) \wedge (q \rightarrow r), \rightarrow (p \rightarrow r)$
T	T	T	T	T	T
T	T	F	F	F	T
T	F	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	T	F	F	T	T
F	F	T	T	T	T
F	F	F	T	T	T

Consider the following argument:

If two sides of a triangle are equal **then**
the opposite angles are equal
T is a triangle with two sides that are not equal

The opposite angles of T are not equal

(With this notation the horizontal line separates a sequence of propositions from a conclusion.)



Let p be the proposition
"two sides of a triangle are equal"
and let q be the proposition
"the opposite angles are equal"

We can re-write the argument in symbols as:

$$p \rightarrow q, \neg p \vdash \neg q$$

and as the expression:

$$[(p \rightarrow q) \wedge \neg p\ ] \rightarrow \neg q$$

We can check that this is a valid argument by using a truth table, and verifying that the expression is a tautology.

p	**q**	$[(p \to q) \wedge \neg p\,] \to \neg q$
T	T	
T	F	
F	T	
F	F	

Propositional Functions

Let P(x) be a propositional function that is either true or false for each x in A.

That is, the domain of P(x) is a set A, and the range is {true, false}. NOTE: Sometimes propositional function are called *predicates*.

Observe that the set A can be partitioned into two subsets:

- Elements with an image that is true.
- Elements with an image that is false.

In particular we may define the *truth set* of P(x) as:
$T_P = \{ x : x \text{ in } A, P(x) \text{ is true}\}$

Examples: Consider the following propositional functions defined on the positive integers.

$P(x)$: $x + 2 > 7$; $T_P = \{x : x > 5\}$
$P(x)$: $x + 5 < 3$; $T_P = \varnothing$
$P(x)$: $x + 5 > 1$; $T_P = \mathbb{N}$

Quantifiers

There are two widely used logical quantifiers

Definition:

Universal Quantifier: $\forall$ (for all)

Let P(x) be a propositional function. A quantified proposition using the propositional function can be stated as:

$(\forall x \in A)\ P(x)$ (for all x in A P(x) is true)

$Tp = \{x : x \in A, P(x)\} = A$

Or if the elements of A can be enumerated as:

$A = \{x_1, x_2, x_3, ...\}$

We would have:

$P(x_1) \wedge P(x_2) \wedge P(x_3) \wedge ...$ is true.

Definition:

Existential Quantifier: $\exists$(there exists)

Let P(x) be a propositional function. A quantified proposition using the propositional function can be stated as:

$(\exists x \in A)\ P(x)$ (There exists an x in A *s.t.* P(x) is true)

$T_P = \{x : x \in A, P(x)\} \neq \varnothing$

Or if the elements of A can be enumerated as:

$A = \{x_1, x_2, x_3, \ldots\}$

We would have:

$P(x_1) \vee P(x_2) \vee P(x_3) \vee \ldots$ is true.

Quantifiers

Statement	True when:	False when:
$(\forall x \in A)\ P(x)$	$P(x)$ is true for every $x \in A$.	$P(x)$ is false for one or more $x \in A$.
$(\exists x \in A)\ P(x)$	$P(x)$ is true for one or more $x \in A$.	$P(x)$ is false for every $x \in A$.

Negating propositions

Let's make this simpler. Let A = {1,2}
Now consider:
$\neg(\exists x \in A)$ such that $2^x < x$

We can expand this by considering every element in A individually as follows

$\neg(2^1 < 1 \vee 2^2 < 2)$

Recall DeMorgan's law (10b) in the "Laws" table.

$\neg (p \vee q) \equiv \neg p \wedge \neg q$

So in this particular case we have:

$$\begin{aligned}\neg(2^1 < 1 \vee 2^2 < 2) &\equiv \neg(2^1 < 1) \wedge \neg(2^2 < 2)\\ &\equiv (2^1 \geq 2) \wedge (2^2 \geq 2)\end{aligned}$$

Proposition: There exists an x in $\mathbb{N}$, such that $2^x < x$.

Let p(x) be the propositional function $2^x < x$
so we have:
$(\exists x \in \mathbb{N})\, p(x)$ (There exists an x in $\mathbb{N}$ *s.t.* p(x) is true)
or
$p(1) \vee p(2) \vee p(3) \vee \ldots$ is true.

And the negation of this logical expression is:

$\neg(p(1) \vee p(2) \vee p(3) \vee \ldots)$

And by extending DeMorgan's law to more than two terms we get

$\neg(p(1) \vee p(2) \vee p(3) \vee \ldots) \equiv (\neg p(1) \wedge \neg p(2) \wedge \neg p(3) \wedge \ldots)$

The right hand side of the congruence can be restated as:

$(\forall x \in \mathbb{N})\, \neg\, p(x)$

Since p(x) is $2^x < x$, we have the negation $\neg\, p(x)$ is $2^x \geq x$

Finally we see that the negation of

“There exists an x in $\mathbb{N}$, such that $2^x < x$”
is:

“For all x in $\mathbb{N}$ $2^x \geq x$.”

And this is an example of the generalized DeMorgan’s Law:

$$\neg(\exists x \in A)p(x) \equiv (\forall x \in A)\neg p(x)$$

and the dual is:

$$\neg(\forall x \in A)p(x) \equiv (\exists x \in A)\neg p(x)$$

Propositional functions with more than one variable

Consider the following illustrative example:

Let p(x,y) be the proposition that "x+y = 10" where the ordered pair $(x,y) \in \{1, 2, ..., 9\} \times \{1, 2, ..., 9\}$.

Consider the following quantified statements:

$$\neg(\forall x \in A)p(x) \equiv (\exists x \in A)\neg p(x)$$

1. $\forall x\ \exists y\ \ p(x,y)$
2. $\exists y\ \forall x\ p(x,y)$

1. Says: "for every x there exists a y such that x + y = 10"
2. Says: "there exists a y such that for every x, x+y = 10"

Statement 1. is true, and statement 2, is false by inspection. This simply illustrates that the concepts that we have seen can be extended to more that one variable.