CISC-102 Winter 2016 Lecture 5

Definition: The set of all (different) subsets of S is the *power set* of S, which we denote as P(S).

If S is a finite set we can prove that:

 $|P(S)| = 2^{|S|}$.

Some examples:

 \emptyset : the empty set has 0 elements, and 1 subset. So $|P(\emptyset)| = 2^0$.

{a}: has 1 element and 2 subsets. So $|P(\{a\})| = 2^1$.

{a,b}: has 2 elements and 4 subsets. So $|P(\{a,b\})| = 2^2$.

{a,b,c}: Let's keep track of the subsets of {a,b,c} by using a binary string counter.



- a b c
- 1 1 1 denotes {a,b,c}
- a b c
- 1 0 1 denotes {a,c}
- a b c
- 0 0 0 denotes \varnothing

We can keep track of the subsets of $\{a,b,c\}$ by using a 0 or 1 in a 3 bit binary string to denote the presence or absence of the symbol in the subset.

For elements of {a,b,c} we say that a is element 1, b is element 2, c is element 3

We also say that the binary bits are numbered 1,2,3 from left to right.

To map a subset to a binary number set bit i to 0 if the element i is not in the subset 1 if the element i is in the subset **Claim:** Every different 3 bit binary string denotes a different subset of {a,b,c}, and every subset of {a, b, c} is represented uniquely by a 3 bit binary string.

If we can count 3 bit binary strings then we can also count subsets of $\{a,b,c\}$.

There are 2 choices for the left bit (bit 1). There are 2 choices for the middle bit (bit 2). There are 2 choices for the right bit (bit 3).

The total number of choices are:

 $2 \times 2 \times 2 = 2^3 = 8.$

This coincides with the claim that:

 $|P(\{a,b,c\})| = 2^3$.

A standard "trick" in mathematics is to obtain a mapping from a new problem to one where the solution is known. When done properly (the mapping has to be one-to-one and onto) the known solution can be used to solve a new problem. **Definition:** Let S be a non-empty set. A *partition* of S consists of disjoint non-empty subsets of S whose union is S.

For example: Let S be the set $\{1,2,3,4,5,6\}$: E = $\{x \in S : x \text{ is even}\}$ O = $\{x \in S : x \text{ is odd}\}$

Observe that $E \cap O = \emptyset$ and $E \cup O = S$, so the set (of sets) $\{E,O\}$ is a partition of S.

This $\{\{1,2\},\{4,3,6\},\{5\}\}$ is another partition of S.

When solving algorithmic problems it is sometimes useful to <u>partition</u> the problem into cases.

A collection or a class of sets can be defined using an index as follows:

Let A_i denote the set $\{x : x \in \mathbb{Z}, x \ge i\}$, for all $i \in \mathbb{N}$. So we have:

 $A_1 = \{ 1, 2, 3, ... \}, A_2 = \{ 2, 3, 4, ... \}, A_3 = \{ 3, 4, 5, ... \}$ etc.

This is a convenient way to define a large class of sets (in this case infinitely many) and leads to some interesting consequences.

1.
$$A_i \subseteq A_j$$
 if $i \ge j$

$$2. \ \bigcup_{i \in \mathbb{N}} A_i = \mathbb{N}$$

$$3. \ \bigcap_{i \in \mathbb{N}} A_i = \emptyset$$

To justify 3. suppose that there is at least one element k in the "intersection". So k must be a positive integer. However by definition there is a set A_{ℓ} such that $\ell > k$ and $k \notin A_{\ell}$. This contradicts the assumption that there is at least one element in the "intersection" and therefore we conclude that the "intersection" is equal to \emptyset .

Principle of Inclusion and Exclusion

Ginger owns a peculiar music store where all of the instruments are either red or if the instrument is not red it is a guitar.

There are 15 red instruments and 18 guitars. How many instruments are there at Ginger's?

Here's a Venn diagram modeling the inventory at Ginger's.



The diagram clearly shows that we also need to know how many of the guitars at Ginger's are red to determine the total number of instruments in the store. So suppose there are 8 red guitars at Ginger's.

Let R denote the set of red instruments at Ginger's and G the set of guitars. We then have:

 $|\mathbf{R}| = 15, |\mathbf{G}| = 18, |\mathbf{R} \cap \mathbf{G}| = 8.$

The total number of instruments is:

 $|\mathbf{R} \cup \mathbf{G}| = |\mathbf{R}| + |\mathbf{G}| - |\mathbf{R} \cap \mathbf{G}| = 15 + 18 - 8 = 25.$

The Principle of Inclusion and Exclusion can be stated as follows:

Theorem: Suppose A and B are finite sets. Then:

 $|A \cup B| = |A| + |B| - |A \cap B|$

This generalizes to a formula for determining the cardinality of the union of three sets.

Corollary: Suppose A, B, and C are finite sets. Then:

 $|A\cup B\cup C|=|A|+|B|+|C| \text{ - } |A\cap B| \text{ - } |A\cap C| \text{ - } |B\cap C|+|A\cap B\cap C|$