CISC-102
Winter 2016
Lecture 6

Principle of Inclusion and Exclusion

The Principle of Inclusion and Exclusion can be stated as follows:

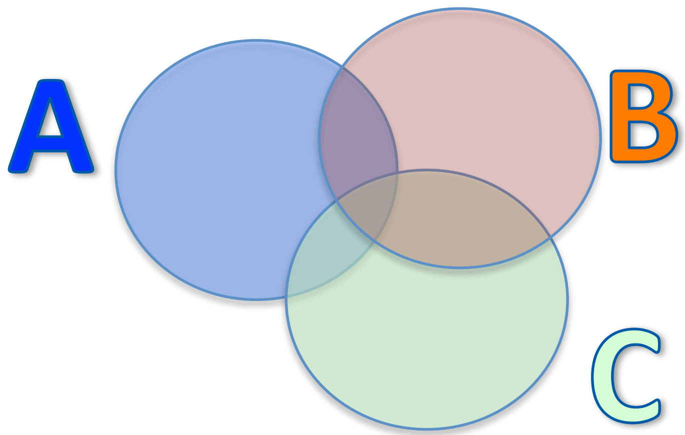
Theorem: Suppose A and B are finite sets. Then:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

This generalizes to a formula for determining the cardinality of the union of three sets.

Corollary: Suppose A, B, and C are finite sets. Then:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$



Consider a collection of 40 people where each of them is wearing something that is red or blue or green such that:

20 wear something blue,
20 wear something red,
20 wear something green
10 wear red and blue, 10 wear red and green, 10 wear blue and green.

How many people in the class are wearing all 3 colours?

Theorem: The proposition P(n), the sum of the first *n* odd numbers is n^2 for all natural numbers *n*.

Preliminaries:

- The k^{th} odd number can be written as $2k-1$.
  e.g.[1] $1 = 2 \times 1-1$, $3 = 2\times 2-1$, $5=2\times 3-1$ *etc*[2].

- **Definition:** A *proposition* is a statement that is either true or false.
  e.g. The earth is flat. The moon is made of cheese.

- A proposition denoted by symbols P(n) are propositions having to do with all numbers of value n.

- **Terminology and notation:** If we say that proposition P(n) is true *for all* natural numbers n, then we mean that: $P(1) \wedge P(2) \wedge P(3) \wedge \ldots$ is true. That is the logical "and" of all of these propositions is true. The symbol $\forall$ is used to denote *for all*.

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[1] exempli gratia are Latin words meaning "for example"

[2] et cetera are Latin words meaning "and so on"

Principle of Mathematical Induction

A *proposition* is defined as a statement that is either true or false. We will at times make a declarative statement as a proposition and then proceed to prove that it is true. Alternately we may provide an example (called a *counterexample*) showing that the proposition is false.

Let P be a proposition defined on the positive integers $\mathbb{N}$; that is, P(n) is either true or false for each $n \in \mathbb{N}$. Suppose P has the following two properties:
(i) P(1) is true.
(ii) P(k+1) is true whenever P(k) is true.

Then by the principle of Mathematical Induction P(n) is true for every positive integer $n \in \mathbb{N}$.

Mathematical induction is by far the most useful tool for proving results in computing.

Note: Step (i) may be replaced by some natural number b > 1 and then the principle of mathematical induction would hold for all natural numbers greater than or equal to b.

Example: The sum of the first n odd numbers is n^2.

Let's try it for some small values of n.
$n = 1$ ($1 = 1^2$), $n = 2$ ($1+ 3 = 4 = 2^2$),
$n = 3$ ($1 + 3 + 5 = 9 = 3^2$)
This is NOT A PROOF! These simply show that the propositions P(1), P(2) and P(3) are true.

This fact will be useful for proving that the sum of the first n odd numbers is n^2.

Theorem: The proposition P(n), the sum of the first *n* odd numbers is n^2 for all natural numbers *n*.

Proof:

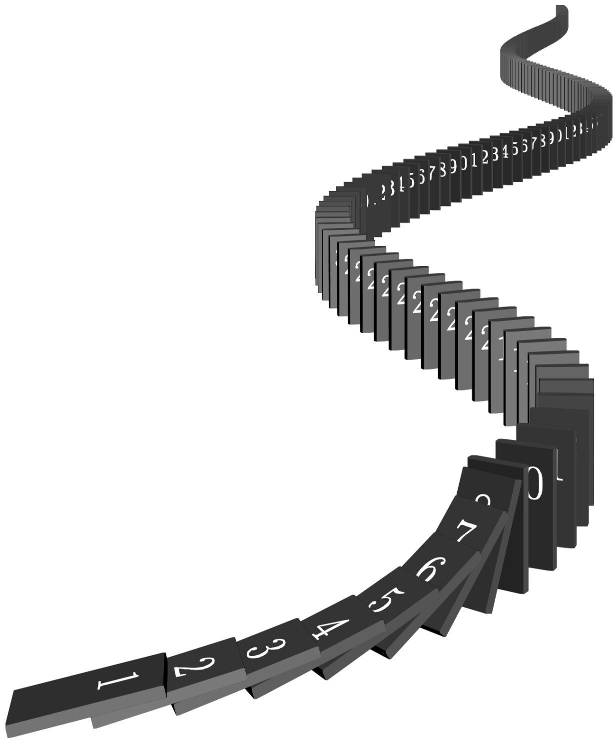
Base case: P(1) $1 = 1^2$, so P(1), the base case is true.

Induction hypothesis: Assume P(k) is true where k is any arbitrary integer greater than or equal to 1.
That is $1 + 3 + 5 + ... + 2k-1 = k^2$.

Induction Step: Consider the sum of the first k+1 odd numbers.

$$\underline{1 + 3 + 5 \ldots + 2k-1} + 2k+1 = \underline{k^2} + 2k+1 \text{ (because P(k) is assumed true)}$$
$$= (k+1)(k+1) \text{ (factor)}$$
$$= (k+1)^2$$

Therefore, we have shown that the proposition P(k) true implies that P(k+1) is true. So by the principle of mathematical induction we conclude that P(n) is true for all natural numbers n. □



As an analogy think of an unending sequence of dominoes. You can be sure that all will fall if:
1. The first one falls. (P(1))
2. And if the k^{th} one falls it will knock over the $k+1^{st}$, that is, P(k) true implies that P(k+1) is also true.

Handshaking revisited: If you recall we observed that the number of handshakes between n people is given by $n(n-1)/2$.
We can make this notion formal by proving it by using mathematical induction.

Preliminaries: Suppose we know the number of handshakes between k people, is the quantity Q, then the number of handshakes for k+1 people is Q + (k), that is, the new person shakes hands with the first k people.

Theorem: The proposition P(n), the number of handshakes for $n \geq 2$ people is $n(n-1)/2$.
Proof:
Base: P(2) For 2 people we have 1 handshake = 2(1)/2 = 1.
Induction Hypothesis: P(k), for k people the number of handshakes is $k(k-1)/2$.
Induction Step: (**Goal:** Show that the number of handshakes with k+1) people is (k+1)(k)/2) For $k+1$ people the number of handshakes is the number of handshakes with k people plus k more handshakes, that is, the $k+1^{st}$ person shakes hands with all the other k people. Therefore using the induction assumption we have:

The number of handshakes with k+1 people is:

$k(k-1)/2 + k$.

We now massage this expression until it fits the statement of the theorem.

$k(k-1)/2 + k = (k^2 - k)/2 + k$ (multiply k(k-1))

$= (k^2 - k + 2k) / 2$ *(obtain common denominator)*

$= (k^2 + k)/2$ *(add -k + 2k)*

$= k(k+1)/2$ *(factor k from $k^2 + k$ to reach goal.)*

Therefore, we have shown that P(k) implies P($k+1$) and by the principle of mathematical induction we conclude that P(n) is true for natural numbers $n \geq 2$. □