CISC-102
Winter 2016
Lecture 7

## Sigma notation

$$\sum_{i=1}^{n} i$$

The big greek letter Sigma is used to represent a sequence of sums. The expression above can be pronounced “sum i for i equal 1 to n. This sum can also be written as:

$$1 + 2 + 3 + ... + n.$$

## Principle of Mathematical Induction

Let P be a proposition defined on the positive integers $\mathbb{N}$; that is, P($n$) is either true or false for each $n \in \mathbb{N}$. Suppose P has the following two properties:
(i) P(1) is true.
(ii) P(k+1) is true whenever P(k) is true.

Then by the principle of Mathematical Induction P($n$) is true for every positive integer $n \in \mathbb{N}$.

**Theorem:** The proposition P(n), the sum of the first *n* odd numbers is $n^2$ for all natural numbers *n*.

**Proof:**

**Base case: P(1)** $1 = 1^2$, so P(1), the base case is true.

**Induction hypothesis:** Assume P(k) is true where k is any arbitrary integer greater than or equal to 1.
That is $1 + 3 + 5 + ... + 2k-1 = k^2$.

**Induction Step: (Goal:** We need to show that the sum of the first k+1 odd numbers is equal to $(k+1)^2$, that is,
$1 + 3 + 5 ... + 2k+1 = (k+1)^2$ )

Consider the sum of the first k+1 odd numbers.

$$\begin{aligned}\underline{1 + 3 + 5 \ldots + 2k-1} + 2k+1 &= \underline{k^2} + 2k+1 \text{ ( because P(k) is assumed true )}\\ &= (k+1)(k+1) \text{ ( factor )}\\ &= (k+1)^2\end{aligned}$$

Therefore, we have shown that the proposition P(k) true implies that P(k+1) is true. So by the principle of mathematical induction we conclude that P(n) is true for all natural numbers n. □

We can rewrite the previous proof using sigma notation.

**Theorem:** The proposition P(n), the sum of the first *n* odd numbers is $n^2$ for all natural numbers *n*.

**Proof:**

**Base case: P(1)** $1 = 1^2$, so P(1), the base case is true.

**Induction hypothesis:** Assume P(k) is true where k is any arbitrary integer greater than or equal to 1. That is,

$$\sum_{i=1}^{k}(2i-1) = k^2$$

**Induction Step: (Goal:** We need to show that the sum of the first k+1 odd numbers is equal to $(k+1)^2$, that is,

$$\sum_{i=1}^{k+1}(2i-1) = (k+1)^2 \quad )$$

Consider the sum of the first k+1 odd numbers.

$$\sum_{i=1}^{k+1}(2i-1) = \sum_{i=1}^{k}(2i-1) + 2k + 1$$

$= k^2 + 2k+1$ ( because P(k) is assumed true )

$= (k+1)(k+1)$ ( factor )

$= (k+1)^2$

Therefore, we have shown that the proposition P(k) true implies that P(k+1) is true. So by the principle of mathematical induction we conclude that P(n) is true for all natural numbers n. □

Let P($n$) be the proposition that a binary string of length $n$ has $2^n$ different values.

**Preliminaries:** The key to proving this result is noticing that adding an additional binary bit to a bit string doubles the total number of values. To see why, assume that you know how many different values can be stored using k bits, numbered 0..k, and collect all those values in the set S. Now consider a k+1 bit string. For every k bit value stored in S we can get two distinct k+1 bit values, one with bit k+1 set to 0 and the other with bit k+1 set to 1.

for example: 8 values using 3 bits are:

000 001 010 011 100 101 110 111

With 4 bits we get 16 values as follows:

0000 0001 0010 0011 0100 0101 0110 0111

1000 1001 1010 1011 1100 1101 1110 1111

**Theorem:** $P(n)$, a binary string of length n stores $2^n$ different values.
Proof:
**Base:** $P(1)$ is true, because 1 bit stores values 0 and 1.
**Induction Hypothesis:** Assume $P(k)$ is true, that is, a binary string with k bits stores $2^k$ different values, for $k \geq 1$.
**Induction Step: (Goal:** Show that $P(k+1)$ is true that is a binary string of length k+1 stores $2^{k+1}$ different values.**)**

By the induction hypothesis we know that a binary string with k bits stores $2^k$ different values.

By the preliminary discussion we saw that adding an additional bit to a binary number doubles the storable values. So we have:

$$( 2^k ) 2 = 2^{k+1}$$

storable values in a binary string with k+1 bits.

Therefore, $P(k)$ implies $P(k+1)$, and by the principle of mathematical induction we conclude that $P(n)$ is true for all natural numbers n. □

A template for proving a theorem by mathematical induction.

Text in **bold** is the same for every proof. The *italicized* text needs to be customized for each proof. Plain text is commentary.

**Base:** Insert the *appropriate base case,* and verify that it is true.
**Induction Hypothesis: Assume that P(k) is true,** that is, insert the *appropriate statement for the proposition P(k).*
**Induction Step: (GOAL:** insert the *appropriate statement for the proposition P(k+1) )* NOTE: Stating the goal explicitly is not a necessity, however, it helps the reader and the writer of the proof keep track of what is to be expected.

Argument that *P(k) true implies that P(k+1) is true*. This is where you need to use your own creativity and technical mathematics ability to attain the stated goal.

**Therefore, P(k) implies P(k+1), and by the principle of mathematical induction we conclude that P(n) is true for** *the appropriate range of values***.**