CISC-102
Winter 2016
Lecture 8

Relations: See chapter 2. of Schaum's

An ordered pair of elements a,b is written as (a,b).

NOTE: Mathematical convention distinguishes between "( )" brackets -- order is important -- and -- "{ }" -- not ordered.

Example: $\{1,2\} = \{2,1\}$, but $(1,2) \neq (2,1)$.

Product sets

Let A and B be two arbitrary sets. The set of all ordered pairs (a,b) where $a \in A$ and $b \in B$ is called the *product* or *Cartesian product*[1] or *cross product* of A and B.

The cross product is denoted as:

$$A \times B = \{(a,b) : a \in A, b \in B \}$$

and is pronounced "A cross B".

It is common to denote $A \times A$ as A^2.

One example of a product set is $\mathbb{R}^2$, that is the product of the Reals, or the two dimensional real plane or Cartesian plane -- x and y coordinates.

[1] Réne Descartes French philosopher mathematician (1596 - 1650)

Examples

Let $A = \{1,2,3\}$, $B = \{4,5,6\}$.

$A \times B = \{(1,4), (1,5), (1,6), (2,4), (2,5), (2,6),$
$(3,4), (3,5), (3,6)\}$.

$\{(1,4),(2,6)\} \subseteq A \times B$.

$(2,5) \in A \times B$.

$\varnothing \subseteq A \times B$.

Let $A = \{1,2,3\}$

$A^2 = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),$
$(3,3)\}$

$\{(1,1),(2,1),(2,3)\} \subseteq A^2.$

$(1,1) \in A^2.$

Relations

Definition: Let A and B be arbitrary sets. A ***binary relation***, or simply a ***relation*** from A to B is a subset of $A \times B$.

Example: Suppose $A = \{1,3,6\}$ and $B = \{1,4,6\}$

$$A \times B = \{(a,b) : a \in A, \text{ and } b \in B\}$$

$$= \{(1,1),(1,4),(1,6),(3,1),(3,4),(3,6),(6,1),(6,4)(6,6)\}$$

Any subset of $A \times B$ is a relation from A to B.

Example: Consider the relation $\leq$ on $A \times B$ where A and B are defined above.

We can define the relation as:

$\{(a,b) : a \in A, \text{ and } b \in B, a \leq b\}$

The subset of $A \times B$ in this relation are the pairs:

$\{(1,1),(1,4),(1,6),(3,4),(3,6),(6,6)\}$

That is, a pair $(a,b) \in A \times B$ is in the relation $\leq$ whenever $a \leq b$.

Vocabulary

When we have a relation on $S \times S$ (which is a very common occurrence) we simply call it a relation *on* S, rather than a relation on $S \times S$.

Let $A = \{1,2,3,4\}$, we can define the following relations on A.

$R_1 = \{(1,1), (1,2), (2,3), (1,3), (4,4)\}$

$R_2 = \{(1,1), (1,2), (2,1), (2,2), (3,3), (4,4)\}$

$R_3 = \{(1,3), (2,1)\}$

$R_4 = \varnothing$

$R_5 = A \times A = A^2$ (How many elements are there in R_5 ?)

Properties of relations on a set A

Reflexive: A relation R is reflexive if $(a,a) \in R$ for all $a \in A$.

Let $A = \{1,2,3,4\}$.

Which of the following relations on A are reflexive?

$R_1 = \{(1,1), (1,2), (2,3), (1,3), (4,4)\}$

$R_2 = \{(1,1), (1,2), (2,1), (2,2), (3,3), (4,4)\}$

$R_3 = \{(1,3), (2,1)\}$

$R_4 = \varnothing$

$R_5 = A \times A = A^2$

Symmetric: A relation R is symmetric if

whenever $(a_1, a_2) \in R$ then $(a_2, a_1) \in R$.

Let $A = \{1,2,3,4\}$.

Which of the following relations on A are symmetric?

$R_1 = \{(1,1), (1,2), (2,3), (1,3), (4,4)\}$

$R_2 = \{(1,1), (1,2), (2,1), (2,2), (3,3), (4,4)\}$

$R_3 = \{(1,3), (2,1)\}$

$R_4 = \varnothing$

$R_5 = A \times A = A^2$

Antisymmetric: A relation R is antisymmetric if

whenever $(a_1, a_2) \in R$ and $(a_2, a_1) \in R$, then $a_1 = a_2$.

Let $A = \{1,2,3,4\}$.

Which of the following relations on A are antisymmetric?

$R_1 = \{(1,1), (1,2), (2,3), (1,3), (4,4)\}$

$R_2 = \{(1,1), (1,2), (2,1), (2,2), (3,3), (4,4)\}$

$R_3 = \{(1,3), (2,1)\}$

$R_4 = \varnothing$

$R_5 = A \times A = A^2$

NOTE: There are relations that are neither symmetric nor antisymmetric or both symmetric and antisymmetric.

$S_1 = \varnothing$ (Both)

$S_2 = \{(1,1), (2,2), (3,3), (4,4)\}$ (Both)

$S3 = \{(1,2),(2,1),(1,3)\}$ (Neither)

Transitive: A relation **R** is transitive if

whenever $(a_1, a_2) \in R$ and $(a_2, a_3) \in R$ then $(a_1, a_3) \in R$.

Let $A = \{1,2,3,4\}$.

Which of the following relations on A are transitive?

$R_1 = \{(1,1), (1,2), (2,3), (1,3), (4,4)\}$

$R_2 = \{(1,1), (1,2), (2,1), (2,2), (3,3), (4,4)\}$

$R_3 = \{(1,3), (2,1)\}$

$R_4 = \varnothing$

$R_5 = A \times A = A^2$

Let A = {1,2,3,4}, we can define the following relations on A.

$R_1 = \{(1,1), (1,2), (2,3), (1,3), (4,4)\}$.

NOT reflexive: (Because (2,2) is missing) .

NOT symmetric: (Because the presence of (1,2) requires (2,1)).

antisymmetric: (No occurrence of a pair, of ordered pairs, of the form (a,b),(b,a)).

transitive: (for every occurrence of the pair $(a_1, a_2) \in R_1$ and $(a_2, a_3) \in R_1$ then $(a_1, a_3) \in R_1$).

$R_2 = \{(1,1), (1,2), (2,1), (2,2), (3,3), (4,4)\}$

reflexive.

symmetric.

NOT antisymmetric.

transitive.

$R_3 = \{(1,3), (2,1)\}$.

NOT reflexive.

NOT symmetric.

antisymmetric.

NOT transitive.

$R_4 = \varnothing$.

NOT reflexive.

symmetric.

antisymmetric.

transitive.

$R_5 = A \times A = A^2$.

reflexive.

symmetric.

NOT antisymmetric.

transitive.

Consider the relation

$R_6 = \{(1,1), (1,2), (2,1), (2,3),(2,2), (3,3)\}$

NOT reflexive:

NOT symmetric:

NOT antisymmetric:

NOT transitive:

Consider the relations $<$, $\leq$, and $=$ on the Natural numbers. (less than, less than or equal to, equal to)

The relation $<$ on the Natural numbers $\{(a,b) : a,b \in N, a < b\}$ is:

NOT reflexive,

NOT symmetric,

antisymmetric,

transitive.

The relation $\leq$ is on the Natural numbers $\{(a,b) : a,b : \in N, a \leq b\}$ is:

reflexive,

NOT symmetric,

antisymmetric,

transitive.

A relation R is called a *partial order* if R is:

reflexive, antisymmetric, and transitive,

so the $\leq$ relation on the natural numbers is a partial order.

The relation = on the Natural numbers $\{(a,b) : a,b : \in N, a = b\}$ is:

reflexive, symmetric, transitive.

A relation R is called an <u>equivalence relation</u> if R is reflexive, symmetric, and transitive, so the = relation on the Natural numbers is an equivalence relation.

Functions

An important special case of a relation, is a function.

A relation from A to B is a *function* if every element $a \in A$ is assigned a unique element of B.

For example: A relation from A to B is ***any*** subset of $A \times B$, any entry in the table below can potentially be an element of a relation, and any entry can be omitted.

	b1	**b2**	**b3**	**b4**	**b5**
a1					
a2					
a3					
a4					

However, a function would require that ***exactly*** one entry per row of the table is present.

Vocabulary

Suppose f is a function from the set A to the set B. Then we say that A is the *domain* of f and B is the *codomain* of f. (Synonyms for codomain are: *target set* & *range*)

Notation

Let f denote a function from A to B, then we write:

$$f : \mathrm{A} \rightarrow \mathrm{B}$$

which is pronounced "f is a function from A to B", or "f maps A into B".

If $a \in$ A, and $b \in$ B we can write:

$$f(a) = b$$

to denote that the function f maps the element a to b.

More Vocabulary

We can say that b is the *image* of a under f.

More notation.

A function can be expressed by a formula (written as an equation, as illustrated by the following example:

$$f(x) = x^2 \text{ for } x \in \mathbb{R}$$

In this example f is the function and x is the variable.

Sometimes we can express the image of a variable (the *independent variable*) by a *dependent variable* as follows:

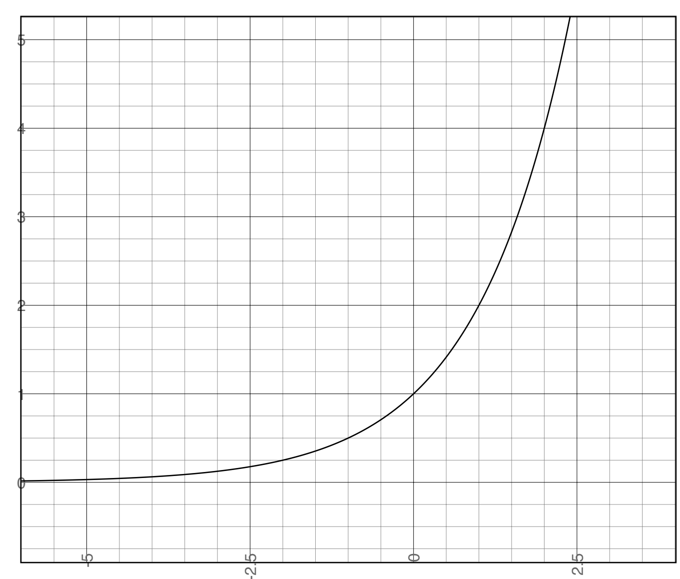
$$y = x^2$$

Injective(one-to-one), Surjective(onto), Bijective(one-to-one and onto) functions.

A function f: A $\rightarrow$ B is a *<u>one-to-one</u>* function if for every a $\in$ A there is a distinct image in B. A one-to-one function is also called an *injective function* or an *injection*.

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ and $f(x) = 2^x$.

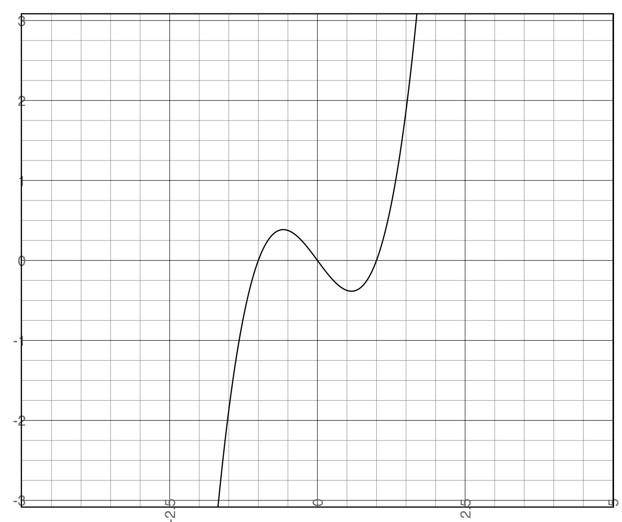
$f(x) = 2^x$ is one-to-one because there is a distinct image for every $x \in \mathbb{R}$, that is if $2^x = 2^y$ then $x = y$.



A function f: A → B is an *onto* function if every $b \in$ B is an image. An onto function is also called a *surjective function* or a *surjection*.

Let $f : \mathbb{R} \to \mathbb{R}$ and $f(x) = x^3 - x$.

$f(x) = x^3 - x$ is onto because the pre-image of any real number y is the solution set of the cubic polynomial equation $x^3 - x - y = 0$ and every cubic polynomial with real coefficients has at least one real root.



Note: $f(x) = x^3 - x = x(x^2 - 1)$ is **not** one-to-one because $f(x) = 0$ for $x = -1, x = +1, x = 0$

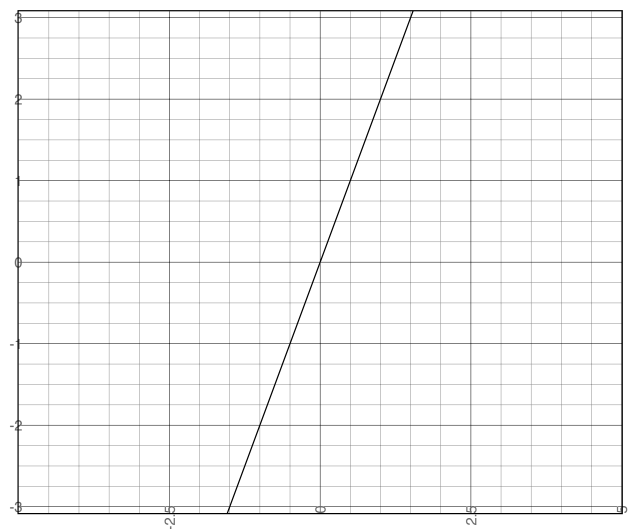
Note: $f(x) = 2^x$ is **not** onto because $2^x > 0$ for all $x \in \mathbb{R}$.

A function that is both one-to-one and onto is called a <u>*bijective function*</u> or a <u>*bijection*</u>.

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ and $f(x) = 2x$

$f(x) = 2x$ is one-to-one because we get a distinct image for every pre-image.

$f(x) = 2x$ is onto because every $y \in \mathbb{R}$ is an image. So $f(x) = 2x$ is a bijection.



Bijective functions are also called *invertible* functions.

That is suppose that f is a bijective function on the set A.

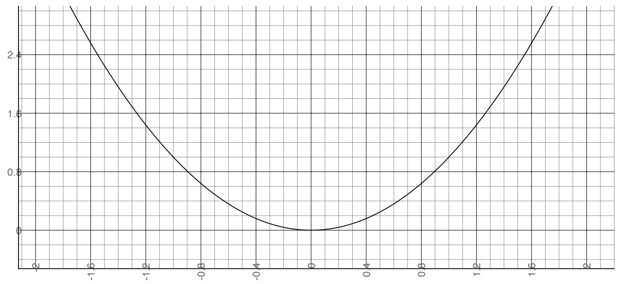
Then f^{-1} denotes the inverse of the function f , meaning that whenever

$f(x) = y$ we have $f^{-1}(y) = x$.

In our previous example we saw that function $f(x) = 2x$ is a bijective function. In this case we can define

$f^{-1}(x) = x/2$, so we get $f^{-1}(2x) = x$.

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ and $f(x) = x^2$



Observe that $f(x) = x^2$ is a function because every $x \in \mathbb{R}$ has a distinct image. However, $f(x) = x^2$ is neither one-to-one (because $f(x) = f(-x)$) or onto ($f(x) \geq 0$).

Composition of functions

Notation: Suppose we have functions $f : A \to B$ and $g : B \to C$, then the composition of f and g written as $g \circ f$ is defined as:

$(g \circ f)(a) = g(f(a))$. (NOTE: carefully notice the order of f and g on the two sides of the equation.)

So for example let $f: \mathbb{R} \to \mathbb{R}$ *be* $f(x) = x^2$ and let $g: \mathbb{R} \to \mathbb{R}$ be $g(x) = 5x$. Then an example of a composition of f and g could be:

$(g \circ f)(2) = g(f(2)) = g(4) = 16$