CISC-102 Winter 2016

Homework 2 Solutions

Problems

1. Illustrate DeMorgan's Law $(A \cap B)^c = A^c \cup B^c$ using Venn diagrams.

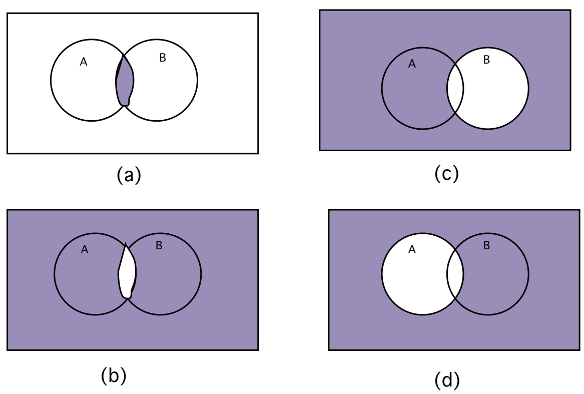

Figure 1: $(A \cap B)$ is shown in (a), and (c) and (d) illustrate B^c and A^c respectively. Finally (b) shows that $(A \cap B)^c = A^c \cup B^c$

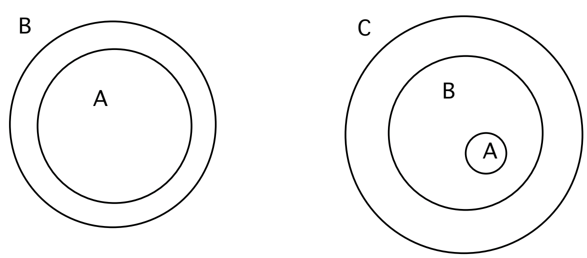

Figure 2: $A \subseteq B$ is shown on the left, and $A \subseteq B \subseteq C$ is shown on the right.

2. Observe that $A \subseteq B$ has the same meaning as $A \cap B = A$. Draw a Venn diagram to illustrate this fact.

See Figure 2. If $A \subseteq B$ then every element $x \in A$ is also an element of B, which in turn implies that $A \cap B = A$.

3. Use a Venn diagram to show that if $A \subseteq B$ **and** $B \subseteq C$, then $A \subseteq C$.

   See Figure 2. $A \subseteq B$ implies that every element of A is also in B, $x \in A$ implies $x \in B$. Similarly $B \subseteq C$ implies that every element of B is also in C, and $y \in B$ implies $y \in C$. Thus $A \subseteq C$.

4. Use the principle of exclusion and inclusion to show that $|A \cup B| + |A \cap B| = |A| + |B|$. (It may help your understanding if you first explore an example such as A = {1,2,3} and B = {3,4}).

   By the principle of exclusion and inclusion we have $|A|+|B|-|A \cap B| = |A \cup B|$. These quantities are non-negative integers (the set of non-negative integers are $\{0\} \cup \mathbb{N}$) so if we subtract $|A \cap B|$ to the right and left side of the equation, we get the desired result, as shown below.

   $$|A \cup B| + |A \cap B| - |A \cap B| = |A| + |B| - |A \cap B|$$

   Now perform the subtraction on the left to get the principle of exclusion and inclusion.

   $$|A \cup B| = |A| + |B| - |A \cap B|$$

5. What are the cardinalities of the following sets?

   (a) A = {winter, spring, summer, fall}. $|A| = 4$.

   (b) B = $\{x : x \in \mathbb{Z}, 0 < x < 7 \}$. $|B| = 6$.

   (c) P(B), that is, the power set of B. $|P(B)| = 2^6 = 64$.

   (d) C = $\{ x : x \in \mathbb{N}, x \text{ is even} \}$. This set has infinitely many elements.

6. Suppose that we have a sample of 100 students at Queen's who take at least one of the following language courses, French-101, Spanish-101, German-101. Also suppose that 65 take French-101, 45 take German-101, 42 take Spanish 101, 20 take French-101 and German-101, 25 take French-101and Spanish-101, and 15 take German-101 and Spanish-101.

   (a) How many students take all three language courses?

   Let F, S, and G denote the sets of students taking French Spanish and German respectively. The principle of inclusion and exclusion tells us that:

$|F \cup S \cup G| = |F| + |S| + |G| - |F \cap S| - |S \cap G| - |F \cap G| + |F \cap S \cap G|$
The problem statement gives us values for each quantity in the equation except for $|F \cap S \cap G|$. We can now simply fill in the numbers and solve for $|F \cap S \cap G|$, as follows:

$$100 = 65 + 42 + 45 - 25 - 15 - 20 + |F \cap S \cap G|$$

So we conclude that $|F \cap S \cap G| = 8$

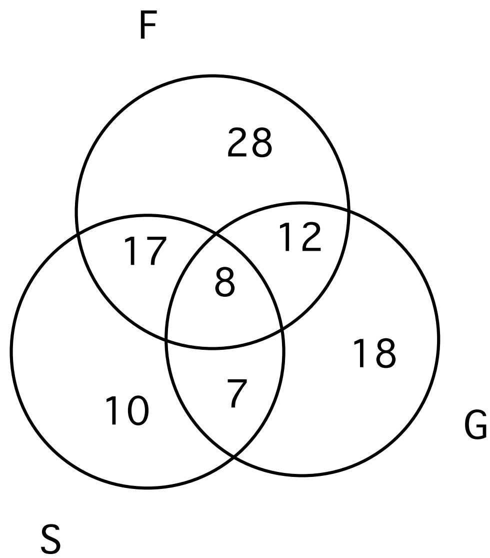

Figure 3: Language Courses Venn Diagram

(b) Draw a Venn diagram representing these 100 students and fill in the regions with the correct number.

(c) How many students take exactly 1 of these courses? Using the Venn diagram we can deduce that 28+10+18 = 56 students take exactly one of the language courses.

(d) How many students take exactly 2 of these courses? Using the Venn diagram we can deduce that 17 +12 +7 = 36 students take exactly two courses.

7. Let S={a,b,c,d,e,f,g}. Determine which of the following are partitions of S:

   (a) P1 = {{a,c,e},{b},{d,g}}. No, because the element f is missing from the union of the sets.

(b) P2 = {{a,b,e,g},{c},{d,f}}. Yes. The union of the sets is S, and the pairwise intersections of the sets are empty.

(c) P3 ={{a,e,g},{c,d},{b,e,f}}. No, because the intersection {a,e,g} $\cap$ {b,e,f} is not the empty set.

(d) P4= {{a,b,c,d,e,f,g}}. Yes, this is technically a partition, but a very uninteresting one.

8. Recall that the union operation is associative, that is $A \cup (B \cup C) = (A \cup B) \cup C$. Show that the relative complement set operation is not associative, that is, $A \backslash (B \backslash C) = (A \backslash B) \backslash C$, is incorrect for some sets A, B, C. (Note if relative complement is associative then the equation must be true for all sets A, B, C.)

   Let A = {1,2,3} B = {1,2} and C = {2,3}. $A \backslash (B \backslash C) = \{2, 3\}$ and $(A \backslash B) \backslash C = \emptyset$.