CISC-102 FALL 2016

HOMEWORK 3 SOLUTIONS

Problems

(1) Let $\{A_i : i \in \mathbb{N}\}$ denote an arbitrary indexed class of sets. Let $k \in \mathbb{N}$ Show that

$$\bigcap_{i \in \mathbb{N}} A_i \subseteq A_k \subseteq \bigcup_{i \in \mathbb{N}} A_i$$

Let

$$A_{int} = \bigcap_{i \in \mathbb{N}} A_i,$$

and observe that it is the intersection of all of the indexed sets.

Therefore, if $x \in A_{int}$ then $x \in A_i$ for all $i \in \mathbb{N}$.

This implies that $A_{int} \subseteq A_i$ for all $i \in \mathbb{N}$, and in particular $A_{int} \subseteq A_k$ for some fixed $k \in \mathbb{N}$.

Let

$$A_{uni} = \bigcup_{i \in \mathbb{N}} A_i.$$

Therefore, if $x \in A_i$ for all $i \in \mathbb{N}$ then $x \in A_{uni}$. This implies that $A_i \subseteq A_{uni}$ for all $i \in \mathbb{N}$, and in particular $A_k \subseteq A_{uni}$ for some fixed $k \in \mathbb{N}$.

(2) Prove using mathematical induction that the sum of the first n natural numbers is equal to $\frac{n(n+1)}{2}$. This can also be stated as:

Prove that the proposition $\mathrm{P}(n)$,

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

is true for all $n \in \mathbb{N}$

Base: for n =1, $1 = \frac{1(1+1)}{2}$

Induction hypothesis: Assume that P(k) is true, that is:

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}.$$

for $k \geq 1$.

Induction step: The goal is to show that P($k+1$) is true, that is:

$$\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}.$$

Consider the sum

$$\begin{aligned}
\sum_{i=1}^{k+1} i &= \sum_{i=1}^{k} i + (k+1) \text{(arithmetic)} \\
&= \frac{k(k+1)}{2} + (k+1) \text{(Use the induction hypothesis)} \\
&= \frac{k^2 + k + 2k + 2}{2} \text{(get common denominator and add)} \\
&= \frac{k^2 + 3k + 2}{2} \text{(add k + 2k)} \\
&= \frac{(k+1)(k+2)}{2} \text{(factor to arrive at goal)}
\end{aligned}$$

Therefore, we have shown that the proposition P(k) true implies that P($k+1$) is true. So by the principle of mathematical induction we conclude that P(n) is true for all natural numbers n. ☐

(3) Prove using mathematical induction that the proposition P(n),

$$\sum_{i=1}^{n} \frac{1}{2^i} = 1 - \frac{1}{2^n},$$

is true, for all $n \in \mathbb{N}$.

Base: $\frac{1}{2} = 1 - \frac{1}{2}$.

Induction hypothesis: Assume that P(k) is true, that is:

$$\sum_{i=1}^{k} \frac{1}{2^i} = 1 - \frac{1}{2^k}.$$

Induction step: The goal is to show that P($k+1$) is true, that is,

$$\sum_{i=1}^{k+1} \frac{1}{2^i} = 1 - \frac{1}{2^{k+1}}.$$

Consider the sum:

$$\begin{aligned}\sum_{i=1}^{k+1} \frac{1}{2^i} =& \sum_{i=1}^{k} \frac{1}{2^i} + \frac{1}{2^{k+1}} \text{(arithmetic)}\\ =&1 - \frac{1}{2^k} + \frac{1}{2^{k+1}} \text{(Use the induction hypothesis)}\\ =&1 + \frac{-2+1}{2^{k+1}} \text{(get common denominator)}\\ =&1 - \frac{1}{2^{k+1}} \text{(add to arrive at goal)}\end{aligned}$$

Therefore, we have shown that the proposition P(k) true implies that P($k+1$) is true. So by the principle of mathematical induction we conclude that P(n) is true for all natural numbers n. □

(4) Prove using mathematical induction that the proposition P(n), the number of values storable in a decimal string (a decimal string uses values, 0, 1, ..., 9) of length n is 10^n.

Base: For $n = 1$ we can store values 0 ... 9 or 10^1 values.

Induction hypothesis: Assume that P(k) is true, that is, we can store 10^k different values in a decimal string of length k.

Induction step: Our goal is to show that we can store 10^{k+1} different values in a decimal string of length $k + 1$.

Observe that when we add one new decimal digit to a decimal string of length k we can realize 10 times the values that we got with a decimal string of length k. Therefore, using the induction hypothesis, we can realize $10 \times 10^k = 10^{k+1}$ different values.

Therefore, we have shown that the proposition $\mathrm{P}(k)$ true implies that $\mathrm{P}(k+1)$ is true. So by the principle of mathematical induction we conclude that $\mathrm{P}(n)$ is true for all natural numbers n. $\square$

(5) Prove using mathematical induction that the proposition $\mathrm{P}(n)$, the number of values storable in a string using k different symbols of length n is k^n.

Base: For $n = 1$ we can store values 0 ... k or k^1 values.

Induction hypothesis: Assume that $\mathrm{P}(j)$ is true, that is, we can store k^j different values in a string of length j. (Note: We are forced to use a different letter , we use " j " , because k is already used.)

Induction step: Our goal is to show that we can store k^{j+1} different values in a string of length $j + 1$ using k different symbols .

Observe that when we add one new symbol to a string using k different symbols of length j we can realize k times the values that we got with a string of length j. Therefore, using the induction hypothesis, we can realize $k \times k^j = k^{j+1}$ different values.

Therefore, we have shown that the proposition $\mathrm{P}(j)$ true implies that $\mathrm{P}(j + 1)$ is true. So by the principle of mathematical induction we conclude that $\mathrm{P}(n)$ is true for all natural numbers n. $\square$