CISC-102 WINTER 2016

HOMEWORK 5 SOLUTIONS

(1) Evaluate

(a) $|3-7| = |-4| = 4$

(b) $|1-4| - |2-9| = |-3| - |-7| = -4$

(c) $|-6-2| - |2-6| = |-8| - |-4| = 4$

(2) Find the quotient q and remainder r, as given by the Division Algorithm theorem for the following examples.

Recall we want to find $r, 0 \leq r < |b|$, such that $a = qb + r$, where all values are integers.

(a) $a = 500, b = 17.$

$500 = 29 \times 17 + 7$ so $r = 7$.

(b) $a = -500, b = 17.$

$-500 = -30 \times 17 + 10$ so $r = 10$.

(c) $a = 500, b = -17.$

$500 = -29 * -17 + 7$ so $r = 7$

(d) $a = -500, b = -17$

$-500 = 30 \times -17 + 10$ so $r = 10$

(3) Show that $c|0$, for all $c \in \mathbb{Z}, c \neq 0$.

$\frac{0}{c} = 0$ for all $c \in \mathbb{Z}, c \neq 0$. Note: $\frac{0}{0}$ is undefined.

(4) Let $a, b, c \in \mathbb{Z}$ such that $c|a$ and $c|b$. Let r be the remainder of the division of b by a, that is there is a $q \in \mathbb{Z}$ such that $b = qa + r, 0 \leq r < |b|$. Show that under these condition we have $c|r$.

Since $c|a$ and $c|b$ we can write:

$$(1) a = cp_a \text{ and } b = cp_b, \text{ such that } p_a, p_b \in \mathbb{Z}.$$

So we can rewrite $b = qa + r$ as:

$$cp_b = qcp_a + r$$

and this simplifies to:

$$c(p_b - qp_a) = r$$

Since $p_b - qb_a$ is an integer we can conclude that $c|r$.

(5) Let $a, b \in \mathbb{Z}$ such that $2|a$. (In other words a is even.) Show that $2|ab$.

This is just a special case of the divisibility theorem that states if $c|a$ then for any integer b, $c|ab$

(6) Let $a \in \mathbb{Z}$ show that $3|a(a+1)(a+2)$, that is the product of three consecutive integers is divisible by 3.

Observe that we can write $a = 3q + r$ where $r \in 0, 1, 2$.

Case 0: If $r = 0$ a is divisible by 3 and since $(a+1)(a+2)$ is an integer it follows that $3|a(a+1)(a+2)$.

Case 1: If $r = 1$, add 2 to both sides of the equation $a = 3q+1$ to get $a+2 = 3q+3 = 3(q+1)$ thus $a+2$ is divisible by 3 and since $a(a+1)$ is an integer it follows that $3|a(a+1)(a+2)$.

Case 2: If $r = 2$, add 1 to both sides of the equation $a = 3q+1$ to get $a+1 = 3q+3 = 3(q+1)$ thus $a+1$ is divisible by 3 and since $a(a+2)$ is an integer it follows that $3|a(a+1)(a+2)$.

(7) Let a be any integer. Let $\mathrm{P}(n)$ denote the proposition:

$$\sum_{i=0}^{n} a^i = \frac{a^{n+1} - 1}{a - 1}$$

Prove that $\mathrm{P}(n)$ is true for all integers $n \geq 0$. Although the first form of induction would suffice to prove this result, use the second form of induction.

Base: $\mathrm{P}(0)$: $a^0 = \frac{a-1}{a-1} = 1$.

Induction Hypothesis: We assume that P(j) is true for $j, 0 \leq j \leq k$, that is, $\sum_{i=0}^{j} a^i = \frac{a^{j+1}-1}{a-1}$ for all $j, 0 \leq j \leq k$.

Induction Step: (Our goal is to show that P($k+1$) is true using the induction hypothesis, that is, $\sum_{i=0}^{k+1} a^i = \frac{a^{k+2}-1}{a-1}$.

$$\begin{aligned}
\sum_{i=0}^{k+1} a^i &= \sum_{i=0}^{k} a^i + a^{k+1} \\
&= \frac{a^{k+1}-1}{a-1} + a^{k+1} \\
&= \frac{a^{k+1}-1+(a-1)a^{k+1}}{a-1} \\
&= \frac{a^{k+1}-1+a^{k+2}-a^{k+1}}{a-1} \\
&= \frac{a^{k+2}-1}{a-1}
\end{aligned}$$

We have proved that P(0) ... P(k) true imply that P(k+1) is true, so by the principle of mathematical induction we conclude that P(n) is true for all integers $n, n \geq 0$ $\square$