CISC-102 WINTER 2016

HOMEWORK 8
SOLUTIONS

(1) How many 5 card hands are there (unordered selection from a standard 52 card deck) that consist of a single pair of the same value, and three other cards of different values? Two possible examples are:

$$\{(2,\heartsuit),(2\diamondsuit),(7\clubsuit),(9\diamondsuit),(3\heartsuit)\} \text{ and } \{(A\heartsuit),(A\clubsuit),(7\diamondsuit),(6\diamondsuit),(3\heartsuit)\}$$

There are $\binom{13}{1}$ ways to choose the value of the pair and $\binom{4}{2}$ ways to choose the suits for the pair. The remaining cards must have three different values from that of the chose pair, so we have $\binom{12}{3}$ possibilities. For each single card there are $\binom{4}{1}$ ways to chose its suit. Therefore we have:

$$\binom{13}{1}\binom{4}{2}\binom{12}{3}\binom{4}{1}^3.$$

(2) How many 5 card hands are there that consist of 5 different values, that are not all the same suit, and not 5 values in sequence. Two possible examples are:

$$\{(8,\heartsuit),(2\diamondsuit),(7\clubsuit),(9\diamondsuit),(3\heartsuit)\} \text{ and } \{(J\heartsuit),(A\clubsuit),(7\diamondsuit),(6\diamondsuit),(3\heartsuit)\}$$

The number of ways of choosing 5 cards of different values is $\binom{13}{5}$. There are $\binom{4}{1}$ ways of choosing the suits of each of these cards.

This yields the expression:

$$\binom{13}{5}\binom{4}{1}^5.$$

However this value is too big because it also counts 5 cards that are in sequence (a straight), and also 5 cards that are all of the suit (a flush).

The number of straights including those that are a straight flush and a royal flush are $\binom{10}{1}\binom{4}{1}^5$. The number of flushes is $\binom{13}{5}\binom{4}{1}$, but this also includes straight flushes, and royal flushes. The total number of straight flushes including royal flushes is $\binom{10}{1}\binom{4}{1}$. So putting this all together we get:

$$\binom{13}{5}\binom{4}{1}^5 - \binom{10}{1}\binom{4}{1}^5 - \binom{13}{5}\binom{4}{1} + \binom{10}{1}\binom{4}{1},$$

and this simplifies to:

$$\binom{13}{5}\binom{4}{1}^4 - \binom{10}{1}\binom{4}{1}^4 = [\binom{13}{5} - \binom{10}{1}]\binom{4}{1}^4.$$

(3) From 100 used cars siting on a lot, 20 are to be selected for a test designed to check safety requirements. These 20 cars will be returned to the lot, and again 20 will be selected for testing for emission standards.

(a) In how many ways can the cars be selected for safety requirement testing?

$$\binom{100}{20}.$$

(b) In how many ways can the cars be selected for emission standards testing?

$$\binom{100}{20}.$$

(c) In how many different ways can the cars be selected for both tests?

$$\binom{100}{20}\binom{100}{20}.$$

(d) In how many ways can the cars be selected for both tests if exactly 5 cars must be tested for safety and emission?

$$\binom{100}{5}\binom{95}{15}\binom{80}{15}.$$

(4) There are 72 students registered in CISC-102 this term. Prove that there is a month of the year in which at least 6 students from this class were born.

If we divide 72 by 12, the number of months of the year, we get 6. Now imagine twelve pigeon holes each containing 5 chairs. Each student in the class enters the pigeon hole representing their month of birth. But since the quotient of 72/12 is larger than 5, we know that there will be at least one pigeon hole where there are more people than chairs.

(5) How many binary strings of length 13 contain precisely 4 1s, and 9 0s? For example 0010101010000 satisfies the requirement and so does 1111000000000.

The number of such binary strings is:

$$\frac{13!}{4!9!} = \binom{13}{4} = \binom{13}{9}.$$

(6) You have 9 identical treats to distribute amongst 5 children.

(a) In how many different ways can the treats be distributed?

We model this type of counting problem by placing 9 identical objects into 5 bins. The bins are obtained by using 4 partition dividers. This can be simulated by a binary string where 0's (9 in this case) represent the objects and 1's (4 of them) represent the partition dividers. Thus we have:

$$\frac{(9+4)!}{4!9!} = \frac{13!}{4!9!} = \binom{13}{4} = \binom{13}{9}.$$

(b) In how many different ways can the treats be distributed if each child must get at least one.

First distributing one treat to each child leaving 4 additional treats. We can solve the problem of distributing 4 identical treats to 5 children as in the previous problem. Thus we have:

$$\frac{(4+4)!}{4!4!} = \frac{8!}{4!4!} = \binom{8}{4}.$$