## CISC-102 WINTER 2016

## HOMEWORK 9 SOLUTIONS

(1) Consider the equation

$$
\begin{equation*}
\binom{12}{5}+\binom{12}{6}=\binom{13}{6} \tag{1}
\end{equation*}
$$

(a) Use a counting argument to prove that the left hand and right hand sides of equation (1) are in fact equal.
On the right we count the number of ways to make an unordered selection of 6 different objects from 13. We can label one of the objects as special and partition the number of choices into those with the special object $\binom{12}{5}$, and those without the special object $\binom{12}{6}$.
(b) Use algebraic manipulation to prove that the left hand and right hand sides of equation (1) are in fact equal.

$$
\begin{aligned}
\binom{12}{5}+\binom{12}{6} & =\frac{12!}{7!5!}+\frac{12!}{6!6!} \\
& =\frac{12!(6)+12!(7)}{7!6!} \\
& =\frac{12!(6+7)}{7!6!} \\
& =\frac{13!}{7!6!} \\
& =\binom{13}{6}
\end{aligned}
$$

(2) The equation

$$
\begin{equation*}
\binom{n-1}{k-1}+\binom{n-1}{k}=\binom{n}{k} \tag{2}
\end{equation*}
$$

generalizes equation (1) above. Use algebraic manipulation to prove that the left hand and right hand sides of equation (2) are in fact equal. Note: A counting argument justifying the equation was given in class.

$$
\begin{aligned}
\binom{n-1}{k-1}+\binom{n-1}{k} & =\frac{(n-1)!}{(k-1)!(n-k)!}+\frac{(n-1)!}{k!(n-k-1)!} \\
& =\frac{(n-1)!(k)+(n-1)!(n-k)}{k!(n-k)!} \\
& =\frac{(n-1)!(k+n-k)}{k!(n-k)!} \\
& =\frac{n!}{k!(n-k)!} \\
& =\binom{n}{k}
\end{aligned}
$$

(3) In the notes for Lecture 16 you will find Pascal's triangle worked out for rows 0 to 8. The numbers in row 8 are 18285670562881 . Work out the values of rows 9 and 10 of Pascal's triangle with the help of equation ( 2 ).

```
row 8: 1 1 8 28 56 70 56 28 8 % 8
row 9: 1 1 9 36 % 84 126 126 84 84 36 9
row10:1 10 44 120
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(4) Pascal's triangle is symmetric about its central column. That is for an odd number of entries in a row (as in row 8) the same numbers are found when moving backward and forward from the central value 70 . A row with an even number of entries such as row 5: 15101051 , exhibits a similar pattern without a unique central value. Explain why Pascal's triangle exhibits this symmetry by using one of the binomial coefficient identities that we saw this week in class.

The identity to use is $\binom{n}{k}=\binom{n}{n-k}$.
A row of Pascal's triangle with an odd number ( $n$ is even) of entries can be viewed as:

$$
\begin{gathered}
\binom{n}{0},\binom{n}{1}, \cdots,\binom{n}{(n / 2)-1}, \\
\binom{n}{(n / 2)},\binom{n}{(n / 2)+1}, \cdots,\binom{n}{n-1},\binom{n}{n}
\end{gathered}
$$

A row of Pascal's triangle with an even number of entries ( $n$ is odd) can be viewed as:

$$
\binom{n}{0},\binom{n}{1}, \cdots,\binom{n}{\lfloor n / 2\rfloor},\binom{n}{\lceil n / 2\rceil}, \cdots,\binom{n}{n-1},\binom{n}{n}
$$

(5) Prove that

$$
\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\binom{n}{3}+\cdots+(-1)^{n}\binom{n}{n}=0
$$

Note that this equation can also be written as follows:

$$
\sum_{i=0}^{n}\binom{n}{i}\left(-1^{i}\right)=0
$$

HINT: This can be viewed as a special case of the binomial theorem.
The binomial theorem with $a=1$ and $b=-1$ can be written as:

$$
0=(1-1)^{n}=\sum_{i=0}^{n}\binom{n}{i}\left(1^{n-i}\right)\left(-1^{i}\right)
$$

And this proves the result.

