CISC-102 Fall 2017

Homework 10
Solutions

1. Prove (using mathematical induction on n) that:

$$\sum_{m=0}^{n} \binom{m+1}{m} = \binom{n+2}{n}$$

is true for all $n \in \mathbb{N}$.

Base: When $n = 1$ we have $\binom{1}{0} + \binom{2}{1} = \binom{3}{1}$

Induction Hypothesis:

$$\sum_{m=0}^{k} \binom{m+1}{m} = \binom{k+2}{k}$$

Induction Step

$$\begin{aligned}\sum_{m=0}^{k+1} \binom{m+1}{m} &= \sum_{m=0}^{k} \binom{m+1}{m} + \binom{k+2}{k+1} \\ &= \binom{k+2}{k} + \binom{k+2}{k+1} \text{(using the induction hypothesis)} \\ &= \binom{k+3}{k+1}\end{aligned}$$

Therefore by the principle of mathematical induction we have shown that

$$\sum_{m=0}^{n} \binom{m+1}{m} = \binom{n+2}{n}$$

is true for all $n \in \mathbb{N}$. □

I will now redo the induction step using $k-1$ for the induction hypothesis and k for the induction step. This makes the arithmetic a bit neater.

Induction Hypothesis:

$$\sum_{m=0}^{k-1} \binom{m+1}{m} = \binom{k+1}{k-1}$$

Induction Step

$$\begin{aligned}
\sum_{m=0}^{k} \binom{m+1}{m} &= \sum_{m=0}^{k-1} \binom{m+1}{m} + \binom{k+1}{k} \\
&= \binom{k+1}{k-1} + \binom{k+1}{k} \text{(using the induction hypothesis)} \\
&= \binom{k+2}{k}
\end{aligned}$$

Therefore by the principle of mathematical induction we have shown that

$$\sum_{m=0}^{n} \binom{m+1}{m} = \binom{n+2}{n}$$

is true for all $n \in \mathbb{N}$. $\square$

2. Use a truth table to verify that the proposition $p \vee \neg(p \wedge q)$ is a tautology, that is, the expression is true for all values of p and q.

p	q	$p \wedge q$	$\neg(p \wedge q)$	$p \vee \neg(p \wedge q)$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

3. Use a truth table to verify that the proposition $(p \wedge q) \wedge \neg(p \vee q)$ is a contradiction, that is, the expression is false for all values of p and q.

p	q	$p \wedge q$	$p \vee q$	$\neg(p \vee q)$	$(p \wedge q) \wedge \neg(p \vee q)$
T	T	T	T	F	F
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	F	T	F

4. Use a truth table to show that $p \vee q \equiv \neg(\neg p \wedge \neg q)$.

p	q	$\neg p$	$\neg q$	$p \vee q$	$\neg p \wedge \neg q$	$\neg(\neg p \wedge \neg q)$
T	T	F	F	T	F	T
T	F	F	T	T	F	T
F	T	T	F	T	F	T
F	F	T	T	F	T	F

5. Show that the following argument is valid.

$$p \to q, \neg q \vdash \neg p$$

We need to show that $[(p \to q) \wedge \neg q] \to \neg p$ is a tautology, and we do so using a truth table as follows:

$\neg p$	p	q	$\neg q$	$p \to q$	$(p \to q) \wedge \neg q$	$[(p \to q) \wedge \neg q] \to \neg p$
F	T	F	T	F	F	T
F	T	T	F	T	F	T
T	F	T	F	T	F	T
T	F	F	T	T	T	T

6. Let A = {1,2,3,4,5}. Determine the truth value of each of the following statements.

   (a) $(\exists x \in A)(x + 2 = 7)$
   This is true with $x = 5$.

   (b) $(\forall x \in A)(x + 2 < 8)$
   This is true, because

   $$(1 + 2 < 8) \wedge (2 + 2 < 8) \wedge (3 + 2 < 8) \wedge (4 + 2 < 8) \wedge (5 + 2 < 8).$$

   (c) $(\exists x \in A)(x + 3 < 2)$
   This is false because:

   $$(1 + 3 \not< 2) \wedge (2 + 3 \not< 2) \wedge (3 + 3 \not< 2) \wedge (4 + 3 \not< 2) \wedge (5 + 3 \not< 2).$$

   (d) $(\forall x \in A)(x + 3 \leq 9)$
   This is true, because

   $$(1 + 3 \leq 9) \wedge (2 + 3 \leq 9) \wedge (3 + 3 \leq 9) \wedge (4 + 3 \leq 9) \wedge (5 + 3 \leq 9).$$

7. Let A = {1,2,3,4,5}. And let $(x, y) \in A^2$, be the domain of the propositions given below. Determine the truth value of the following statements.

(a) $\exists x \forall y, x^2 < y + 1$
The statement is true because $1^2 < y + 1$ for every $y \in A$.

(b) $\forall x \exists y, x^2 < y + 1$
The statement is false because there is no $y \in A$ such that $5^2 < y + 1$.