CISC-102 Winter 2017

Homework 2
Solutions

January 23, 2017

Problems

1. Illustrate DeMorgan's Law $(A \cap B)^c = A^c \cup B^c$ using Venn diagrams.

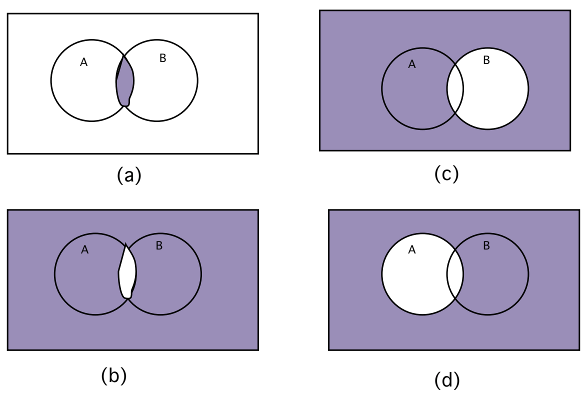

Figure 1: $(A \cap B)$ is shown in (a), and (c) and (d) illustrate B^c and A^c respectively. Finally (b) shows that $(A \cap B)^c = A^c \cup B^c$

2. Let $A_i = \{1, 2, 3, \ldots, i\}$ for all $i \in \mathbb{N}$. For example $A_4 = \{1, 2, 3, 4\}$.

   What are the elements of the set:

   (a) What are the elements of the set $\cup_{i=1}^{n} A_i$?

   $$\bigcup_{i=1}^{n} A_i = \{1, 2, \ldots, n\}$$

   (b) What are the elements of the set $\cap_{i=1}^{n} A_i$?

   $$\bigcap_{i=1}^{n} A_i = \{1\}$$

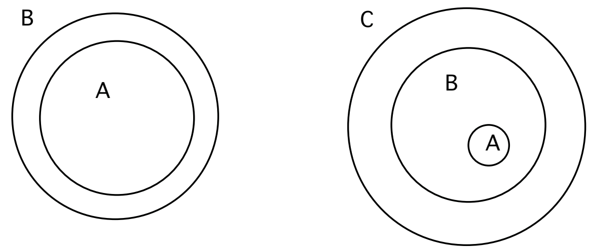

Figure 2: $A \subseteq B$ is shown on the left, and $A \subseteq B \subseteq C$ is shown on the right.

3. Observe that $A \subseteq B$ has the same meaning as $A \cap B = A$. Draw a Venn diagram to illustrate this fact.

   See Figure 2. If $A \subseteq B$ then every element $x \in A$ is also and element in B, which in turn implies that $A \cap B = A$.

4. Use a Venn diagram to show that if $A \subseteq B$ **and** $B \subseteq C$, then $A \subseteq C$.

   See Figure 2. $A \subseteq B$ implies that every element of A is also in B, $x \in A$ implies $x \in B$. Similarly $B \subseteq C$ implies that implies that every element of B is also in C, $y \in B$ implies $y \in C$. Thus $A \subseteq C$.

5. Use the Principle of Exclusion and Inclusion to show that $|A \cup B| + |A \cap B| = |A| + |B|$. (It may help your understanding if you first explore an example such as A = {1,2,3} and B = {3,4}).

   By the Principle of Inclusion Exclusion we have $|A| + |B| - |A \cap B| = |A \cup B|$. These quantities are just non-negative integers so if we add $|A \cap B|$ to the right and left side of the equation, we get the desired result.

6. What are the cardinalities of the following sets?

   (a) A = {winter, spring, summer, fall}. $|A| = 4$.

   (b) B = $\{x : x \in \mathbb{Z}, 0 < x < 7\}$. $|B| = 6$.

   (c) P(B), that is, the power set of B. $|P(B)| = 2^6 = 64$.

   (d) C = $\{ x : x \in \mathbb{N}, x \text{ is even} \}$ This set has infinitely many elements.

7. Suppose that we have a sample of 100 students at Queen's who take at least one of the following language courses, French-101, Spanish-101, German-101. Also suppose that 65 take French-101, 45 take German-101, 42 take Spanish 101, 20 take French-101 and German-101, 25 take French-101and Spanish-101, and 15 take German-101 and Spanish-101.

   (a) How many students take all three language courses?

Let F, S, and G denote the sets of students taking French Spanish and German respectively. The Principle of Inclusion and Exclusion tells us that $|F \cup S \cup G| = |F| + |S| + |G| - |F \cap S| - |S \cap G| - |F \cap G| + |F \cap S \cap G|$
The problem statement gives us values for each quantity in the equation except for $|F \cap S \cap G|$. We can now simply fill in the numbers and solve for $|F \cap S \cap G|$, as follows:

$$100 = 65 + 42 + 45 - 25 - 15 - 20 + |F \cap S \cap G|$$

So we conclude that $|F \cap S \cap G| = 8$

(b) Draw a Venn diagram representing these 100 students and fill in the regions with the correct number.

(c) How many students take exactly 1 of these courses? Using the Venn diagram we can deduce that 28+10+18 = 56 students take exactly one of the language courses.

(d) How many students take exactly 2 of these courses? Using the Venn diagram we can deduce that 17 +12 +7 = 36 students take exactly two courses.

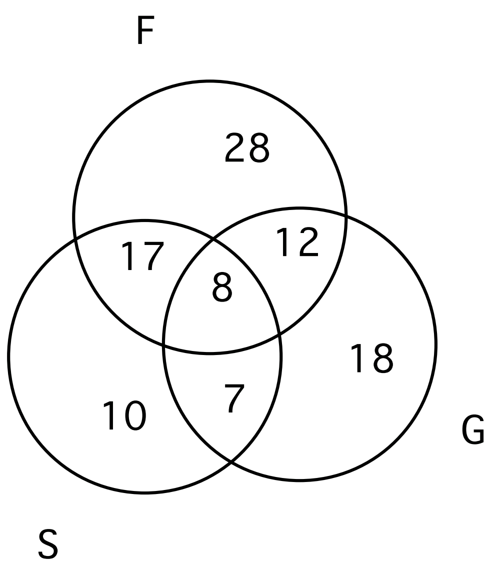

Figure 3: Language Courses Venn Diagram

8. Let S={a,b,c,d,e,f,g}. Determine which of the following are partitions of S:

   (a) P1 = [{a,c,e},{b},{d,g}] No, because f is missing from the union of the sets.

(b) P2 = [{a,b,e,g},{c},{d,f}] Yes. The union of the sets is S, and the pairwise intersections of the sets are empty.

(c) P3 =[{a,e,g},{c,d},{b,e,f}] No, because the intersection of $\{a, e, g\} \cap \{b, e, f\}$ is not empty.

(d) P4= [{a,b,c,d,e,f,g}] Yes, this is technically a partition, but a very uninteresting one.

9. Recall that the union operation is associative, that is $A \cup (B \cup C) = (A \cup B) \cup C$. Show that the relative complement set operation is not associative, that is, $A \backslash (B \backslash C) = (A \backslash B) \backslash C$, is incorrect for some sets A, B, C. (Note if relative complement is associative then the equation must be true for all sets A, B, C.)

   Let A = {1,2,3} B = {1,2} and C = {2,3}. $A \backslash (B \backslash C) = \{2, 3\}$ and $(A \backslash B) \backslash C = \emptyset$

10. Consider a set S of n elements, such that $\{a, b\} \subseteq S$.

   (a) What is the cardinality of the power set of $S \setminus \{a\}$?

   We know that S has n elements so $S \setminus \{a\}$ has $n - 1$ elements. The power set of $S \setminus \{a\}$ has 2^{n-1} elements.

   (b) What is the cardinality of the power set of $S \setminus \{a, b\}$?

   The power set of $S \setminus \{a, b\}$ has 2^{n-2} elements.

   (c) How many subsets of S are there that contain the element a?

   Here is a way to construct the subsets of S that contain the element a. For each subset $s \in P(S \setminus \{a\})$ construct the set $\{a\} \cup s$. This yields all subsets of S that contain a. Since there are 2^{n-1} subsets of $S \setminus \{a\}$, there are 2^{n-1} subsets of S that contain the element a.

   We can obtain the same result by using a different argument. We know that there are 2^n subsets of S. The subsets of $S \setminus \{a\}$ are also subsets of S that do not contain a. The total number of subsets of S is 2^n. So the number of subsets of S that contain a is equal to $2^n - 2^{n-1} = 2^{n-1}$.

   (d) How many subsets of S are there that contain the element a and exclude the element b?

   Here is a way to construct the subsets of S that contain a and exclude b. For each subset $s \in P(S \setminus \{a, b\})$ construct the set $\{a\} \cup s$. This yields 2^{n-2} subsets of S.