CISC-102 FALL 2017

HOMEWORK 3
SOLUTIONS

(1) Prove using mathematical induction that the sum of the first n natural numbers is equal to $\frac{n(n+1)}{2}$. This can also be stated as:

Prove that the proposition P(n),

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

is true for all $n \in \mathbb{N}$.

Base: for n =1, $1 = \frac{1(1+1)}{2}$

Induction hypothesis: Assume that P(k) is true, that is:

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}.$$

for $k \geq 1$.

Induction step: The goal is to show that P$(k+1)$ is true, that is:

$$\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}.$$

Consider the sum

$$\begin{aligned}\sum_{i=1}^{k+1} i =& \sum_{i=1}^{k} i + (k+1)\text{(arithmetic)}\\ =& \frac{k(k+1)}{2} + (k+1)\text{(Use the induction hypothesis)}\\ =& \frac{k^2+k+2k+2}{2}\text{(get common denominator and add)}\\ =& \frac{k^2+3k+2}{2}\text{(add k + 2k)}\\ =& \frac{(k+1)(k+2)}{2}\text{(factor to arrive at goal)}\end{aligned}$$

We have shown that P(k) true implies that P$(k+1)$ is true so by the principle of mathematical induction we conclude that P(n) is true for all $n \in \mathbb{N}$. □

(2) Prove using mathematical induction that the proposition P(n),

$$\sum_{i=2}^{n} i = \frac{(n-1)(n+2)}{2}$$

is true for all $n \in \mathbb{N}, n \geq 2$

Base: for $n = 2$, $2 = \frac{1(2+2)}{2}$

Induction hypothesis: Assume that P(k) is true, that is:

$$\sum_{i=2}^{k} i = \frac{(k-1)(k+2)}{2}.$$

for $k \geq 2$.

Induction step: The goal is to show that P($k+1$) is true, that is:

$$\sum_{i=2}^{k+1} i = \frac{(k)(k+3)}{2}.$$

Consider the sum

$$\sum_{i=2}^{k+1} i = \sum_{i=2}^{k} i + (k+1)\text{(arithmetic)}$$
$$= \frac{(k-1)(k+2)}{2} + (k+1)\text{(Use the induction hypothesis)}$$
$$= \frac{k^2+k-2+2k+2}{2}\text{(get common denominator and add)}$$
$$= \frac{k^2+3k}{2}\text{(arithmetic)}$$
$$= \frac{k(k+3)}{2}\text{(factor to arrive at goal)}$$

We have shown that P(k) true implies that P($k+1$) is true so by the principle of mathematical induction we conclude that P(n) is true for all $n \in \mathbb{N}, n \geq 2$. □

(3) Prove using mathematical induction that the proposition P(n),

$$\sum_{i=3}^{n} i = \frac{(n-2)(n+3)}{2}$$

is true for all $n \in \mathbb{N}, n \geq 3$

Base: for $n = 3$, $3 = \frac{(3-2)(3+3)}{2}$

Induction hypothesis: Assume that P(k) is true, that is:

$$\sum_{i=3}^{k} i = \frac{(k-2)(k+3)}{2}.$$

for $k \geq 3$.

Induction step: The goal is to show that P$(k+1)$ is true, that is:

$$\sum_{i=3}^{k+1} i = \frac{(k-1)(k+4)}{2}.$$

Consider the sum

$$\begin{aligned}
\sum_{i=3}^{k+1} i &= \sum_{i=3}^{k} i + (k+1) \text{(arithmetic)} \\
&= \frac{(k-2)(k+3)}{2} + (k+1) \text{(Use the induction hypothesis)} \\
&= \frac{k^2 + k - 6 + 2k + 2}{2} \text{(get common denominator and add)} \\
&= \frac{k^2 + 3k - 4}{2} \text{(arithmetic)} \\
&= \frac{(k-1)(k+4)}{2} \text{(factor to arrive at goal)}
\end{aligned}$$

We have shown that P(k) true implies that P($k+1$) is true so by the principle of mathematical induction we conclude that P(n) is true for all $n \in \mathbb{N}, n \geq 3$. □

(4) Prove using mathematical induction that the proposition P(n)

$$n! \leq n^n$$

is true for all $n \in \mathbb{N}$.

Base: for n =1, $1! = 1 = 1^1$

Induction hypothesis: Assume that P(k) is true, that is:

$$k! \leq k^k$$

for $k \geq 1$.

Induction step: The goal is to show that P($k+1$) is true, that is:

$$(k+1)! \leq (k+1)^{k+1}.$$

We have:

$$\begin{aligned}(k+1)! &= k!(k+1) \text{(Definition of factorial)}\\ &\leq k^k(k+1) \text{(Use the induction hypothesis)}\\ &\leq (k+1)^k(k+1) \text{(because } k \leq k+1\text{)}\\ &= (k+1)^{k+1} \text{(multiply)}\end{aligned}$$

We have shown that P(k) true implies that P($k+1$) is true so by the principle of mathematical induction we conclude that P(n) is true for all $n \in \mathbb{N}$. □