CISC-102 FALL 2017

HOMEWORK 5 SOLUTIONS

(1) Consider the following relations on the set $A = \{1, 2, 3\}$:

- $R = \{(1, 1), (1, 2), (1, 3), (3, 3)\}$,
- $S = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$,
- $T = \{(1, 1), (1, 2), (2, 2), (2, 3)\}$,
- $A \times A$.

For each of these relations determine whether it is symmetric, antisymmetric, reflexive, or transitive.

S and A × A are symmetric.

R and T are antisymmetric.

S and A × A are reflexive.

R, S and A × A are transitive.

(2) Explain why each of the following binary relations on the set $S = \{1, 2, 3\}$ is or is not an equivalence relation on S.

(a) $R_1 = \{(1, 1), (1, 2), (3, 2), (3, 3), (2, 3), (2, 1)\}$
(b) $R_2 = \{(1, 1), (2, 2), (3, 3), (2, 1), (1, 2),$
$(3, 2), (2, 3), (3, 1), (1, 3)\}$
(c) $R_3 = \{(1, 1), (2, 2), (3, 3), (3, 1), (1, 3)\}$

R_1, is neither reflexive nor transitive so it's not an equivalence relation. R_1 is symmetric.

R_2 is reflexive, symmetric, and transitive so it is an equivalence relation.

R_3 is reflexive, symmetric and transitive, so it is an equivalence relation.

(3) Let R be a relation on the set of Natural numbers such that $(a, b) \in$ R if $a \geq b$. Show that the relation R on $\mathbb{N}$ is a partial order.

R is reflexive because for all $a \in (N)$ $a \geq a$. R is antisymmetric because for all $a, b \in \mathbb{N}, a \neq b$ we have either $a \geq b$ or $b \geq a$ but not both. R is transitive because for all $a, b, c \in \mathbb{N}$, if $a \geq b$ and $b \geq c$, we have $a \geq c$.

(4) Evaluate

(a) $|3-7| = |-4| = 4$

(b) $|1-4| - |2-9| = |-3| - |-7| = -4$

(c) $|-6-2| - |2-6| = |-8| - |-4| = 4$

(5) Find the quotient q and remainder r, as given by the Division Algorithm theorem for the following examples.

Recall we want to find $r, 0 \leq r < |b|$, such that $a = qb + r$, where all values are integers.

(a) $a = 500, b = 17.$
$500 = 29 \times 17 + 7$ so $r = 7.$

(b) $a = -500, b = 17.$
$-500 = -30 \times 17 + 10$ so $r = 10.$

(c) $a = 500, b = -17.$
$500 = -29 \times -17 + 7$ so $r = 7$

(d) $a = -500, b = -17$
$-500 = 30 \times -17 + 10$ so $r = 10$

(6) Show that $c|0$, for all $c \in \mathbb{Z}, c \neq 0.$

Recall the definition of divisibility:

If $c = \frac{b}{a}$ is an integer, or alternately if c is an integer such that $b = ca$ then we say that a divides b or equivalently, b is divisible by a, and this is written $a|b$.

Since $\frac{0}{c} = 0$ for all $c \in \mathbb{Z}, c \neq 0$, and 0 is an integer we have shown that every integer c divides 0. Note: $\frac{0}{0}$ is undefined.

(7) Let $a, b, c \in \mathbb{Z}$ such that $c|a$ and $c|b$. Let r be the remainder of the division of b by a, that is there is a $q \in \mathbb{Z}$ such that $b = qa + r, 0 \leq r < |b|$. Show that under these condition we have $c|r$.

Since $c|a$ and $c|b$ we can write:

$$(1) a = cp_a \text{ and } b = cp_b, \text{ such that } p_a, p_b \in \mathbb{Z}.$$

So we can rewrite $b = qa + r$ as:

$$cp_b = qcp_a + r$$

and this simplifies to:

$$c(p_b - qp_a) = r$$

Since $p_b - qp_a$ is an integer we can conclude that $c|r$.

(8) Let $a, b \in \mathbb{Z}$ such that $2|a$. (In other words a is even.) Show that $2|ab$.

This is just a special case of the divisibility theorem that states if $c|a$ then for any integer b, $c|ab$

(9) Let $a \in \mathbb{Z}$ show that $3|a(a+1)(a+2)$, that is the product of three consecutive integers is divisible by 3.

Observe that we can write $a = 3q + r$ where $r \in \{0, 1, 2\}$.

Case 0: If $r = 0$ a is divisible by 3 and since $(a+1)(a+2)$ is an integer it follows that $3|a(a+1)(a+2)$.

Case 1: If $r = 1$, add 2 to both sides of the equation $a = 3q + 1$ to get $a + 2 = 3q + 3 = 3(q + 1)$ thus $a + 2$ is divisible by 3 and since $a(a + 1)$ is an integer it follows that $3|a(a + 1)(a + 2)$.

Case 2: If $r = 2$, add 1 to both sides of the equation $a = 3q + 2$ to get $a + 1 = 3q + 3 = 3(q + 1)$ thus $a + 1$ is divisible by 3 and since $a(a + 2)$ is an integer it follows that $3|a(a + 1)(a + 2)$.

(10) Use induction to prove the following propositions.

(a) Use induction to prove $n^3 + 2n$ is divisible by 3, for all $n \in \mathbb{N}, n \geq 1$.
Base: $3|1^3 + 2$
Induction Hypothesis: Assume that $k^3 + 2k$ is divisible by 3, for $k \geq 1$.
Induction Step: Goal: Show that $3|(k + 1)^3 + 2(k + 1)$ using the induction hypothesis.
We begin by manipulating the expression $(k + 1)^3 + 2(k + 1)$ as follows:

$$\begin{aligned}(k+1)^3 + 2(k+1) &= k^3 + 3k^2 + 3k + 1 + 2k + 2 \\ &= k^3 + 2k + 3(k^2 + k + 1)\end{aligned}$$

Observe that $3|k^3 + 2k$ by the induction hypothesis and $3|3(k^2 + k + 1)$. So $3|k^3 + 2k + 3(k^2 + k + 1)$.
Therefore by the principle of mathematical induction we conclude that n^3+2n is divisible by 3, for all $n \in \mathbb{N}, n \geq 1$. □

(b) Show that any integer value greater than 2 can be written as $3a + 4b + 5c$, where a, b, c are non-negative integers, that is $a, b, c \in \mathbb{Z}, a, b, c \geq 0$.

We use the 2nd form of induction to prove this result.

Base: We use three base cases: $3 = 3\times1+4\times0+5\times0, 4 = 3\times0+4\times1+5\times0, 5 = 3 \times 0 + 4 \times 0 + 5 \times 1$.

Induction Hypothesis: Assume that all values j such that $2 \leq j \leq k$ can be written as $3a + 4b + 5c$, where a, b, c are non-negative integers.

Induction Step:

By the induction hypothesis we can write $k = 3a + 4b + 5c$. There are three cases to consider:

$a > 0$ (Note: Using 3 as a base is an example of this case.)

Since $k = 3a+4b+5c$ and $a > 0$. We can write $k+1 = 3(a-1)+4(b+1)+5c$.

$a = 0, b > 0$ (Note: Using 4 as a base is an example of this case.)

Since $k = 4b + 5c$ and $b > 0$. We can write $k + 1 = 4(b - 1) + 5(c + 1)$.

$a = 0, b = 0, c > 0$ (Note: Using 5 as a base is an example of this case.)

Since $k = 5c$ and $c > 0$ We can write $k + 1 = 3 \times 2 + 5(c - 1)$.

Therefore, by the principle of mathematical induction we conclude that any integer value greater than 2 can be written as $3a + 4b + 5c$, where a, b, c are non-negative integers. $\square$

(c) Show that every Natural number n can be represented as a sum of distinct powers of 2. For example the number $42 = 32 + 8 + 2 = 2^5 + 2^3 + 2^1$.

We use the second form of induction to prove this result.

Base: $1 = 2^0$.

Induction Hypothesis: Assume that all values j can be represented as a sum of distinct powers of 2, for $1 \leq j \leq k$.

Induction Step: Consider the number $k + 1$. Let 2^a be the largest power of 2 less than or equal to $k + 1$. Now let $b = k + 1 - 2^a$. If $b = 0$ we are done. Otherwise observe that $b \leq k$, and by the induction hypothesis b can be represented as a sum of distinct powers of 2. This, leads to the conclusion that $k + 1$ is also represented as a sum of distinct powers of 2.

Therefore, by the principle of mathematical induction we conclude that every Natural number n can be represented as a sum of distinct powers of 2 $\square$