CISC-102 FALL 2017

HOMEWORK 9 SOLUTIONS

(1) A skip straight is 5 cards that are in consecutive order, skipping every second rank (for example 3-5-7-9-J). How many 5 card hands are there (unordered selection from a standard 52 card deck) that form a skip straight?

Observe that once the lowest value of the skip straight is selected the values of the 4 remaining cards are pre-determined. The highest the lowest value of a skip straight can be is 6, because 6-8-10-Q-A is a valid skip straight, but if we start at 7 or higher we run out of values at the high end. This leads us to conclude that there are 6, or $\binom{6}{1}$ ways to get a skip straight without concern for the suit of each card. There are 4, or $\binom{4}{1}$ ways to select the suit for each card in the skip straight.

Putting this all together we get the product:

$$\binom{6}{1}\binom{4}{1}^5.$$

Note: This also counts the case where all 5 cards are of the same suit. We can call this a *skip straight flush*. The number of straight skip flushes is:

$$\binom{6}{1}\binom{4}{1}.$$

(2) Let S be a finite subset of the positive integers. What is the smallest value for $|S|$ that guarantees that at least two elements $x, y \in S$ have the same remainder when divided by 100. HINT: Use the pigeon hole principle.

There are 100 residue classes (mod 100) so by the pigeon hole principle any subset of the positive integers that has at least 101 elements has two or more elements that are in the same residue class, or equivalently have the same remainder when divided by 100.

(3) Prove that any set of 5 natural numbers will always have two numbers n_1 and n_2 such that $4|(n_1 - n_2)$. Hint: Use the Pigeon Hole Principle.

We know that $4|(n_1 - n_2)$ implies that n_1 and n_2 have the same remainder when we divide by 4, or equivalently are in the same residue class (mod 4). So by the pigeon hole principle any set of 5 natural numbers will have at least two numbers in the same residue class.

(4) Use the binomial theorem to expand the product $(x + y)^6$.

Recall: The binomial theorem can be stated as:

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i.$$

So for this question we have:

$$(x+y)^6 = \binom{6}{0}x^6 + \binom{6}{1}x^5 y + \binom{6}{2}x^4y^2 + \binom{6}{3}x^3y^3 + \binom{6}{4}x^2y^4 + \binom{6}{5}xy^5 + \binom{6}{6}y^6.$$

(5) Show that

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + \binom{n}{n} = 0$$

HINT: Use the Binomial theorem.
Note that this equation can also be written as follows:

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i = 0$$

The binomial theorem with $a = 1$ and $b = -1$ can be written as:

$$0 = (1-1)^n = \sum_{i=0}^{n} \binom{n}{i} (1^{n-i})(-1)^i$$

And this proves the result.