CISC-102
Fall 2017
Week 10

Pascal's Triangle

An easy way to calculate a table of binomial coefficients was recognized centuries ago by mathematicians in India, China, Iran and Europe. In the west the technique is named after the French mathematician Blaise Pascal (1623-1662). In the example below each row represents the binomial coefficients as used in the binomial theorem.

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$
$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$
$$\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}$$

To obtain the entries by hand in a simple way we can use the identity:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$
$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$
$$\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}$$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

Consider the sum of elements in a row of Pascal's triangle. If we label the top row 0, then it appears that row i sums to the value 2^i. Can you explain why this is the case?

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$
$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$
$$\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}$$

$$1$$
$$1 \quad 1$$
$$1 \quad 2 \quad 1$$
$$1 \quad 3 \quad 3 \quad 1$$
$$1 \quad 4 \quad 6 \quad 4 \quad 1$$
$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$
$$1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1$$
$$1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1$$
$$1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1$$

Now let's compute the sum of squares of the entries of each row in Pascal's triangle.

$1^2 = 1$
$1^2 + 1^2 = 2$
$1^2 + 2^2 + 1^2 = 6$
$1^2 + 3^2 + 3^2 + 1^2 = 20$
$1^2 + 4^2 + 6^2 + 4^2 + 1^2 = 70$

These sums all appear in the middle row of Pascal's triangle.

$$\begin{array}{ccccccccccccccccc}
&&&&&&&&1&&&&&&&& \\
&&&&&&&1&&1&&&&&&& \\
&&&&&&1&&2&&1&&&&&& \\
&&&&&1&&3&&3&&1&&&&& \\
&&&&1&&4&&6&&4&&1&&&& \\
&&&1&&5&&10&&10&&5&&1&&& \\
&&1&&6&&15&&20&&15&&6&&1&& \\
&1&&7&&21&&35&&35&&21&&7&&1& \\
1&&8&&28&&56&&70&&56&&28&&8&&1
\end{array}$$

Which leads us to conjecture that:

$$\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}$$

Before proving the theorem there are two preliminary lemmas.

Lemma 1:

$$\binom{n}{k}\binom{n}{n-k} = \binom{n}{k}^2$$

For all non-negative integers *n,k, n* > *k*.

Proof: Since we already showed that $\binom{n}{k} = \binom{n}{n-k}$ this should be obvious. □

$$\sum_{i=0}^{k} \binom{m}{k-i}\binom{n}{i} = \binom{m+n}{k}$$

Lemma 2:

For all non-negative integers *m,n,k* such that $n \geq m \geq k$.

Proof: We use a counting argument. The right hand side can be viewed as the number of subsets of size *k* chosen from the union of two disjoint sets, *S* of size *m*, and *T* of size *n*. On the left we sum the choices where all *k* are from *S*, then *k-1* from *S* and 1 from *T* and so on up to all *k* chosen from set *T*. □

For example: Suppose
S = {a,b} with |S| = *m* = 2, and
T = {c,d,e} with |T| = *n* = 3 and
k = 2. So the sum on the right would be:

$$\sum_{i=0}^{2} \binom{2}{2-i}\binom{3}{i} = \binom{2}{2}\binom{3}{0} + \binom{2}{1}\binom{3}{1} + \binom{2}{0}\binom{3}{2}$$

Theorem:

$$\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}$$

for all natural numbers n ≥ 1.

Proof: Using lemma 1 we can write $\binom{n}{i}^2 = \binom{n}{i}\binom{n}{n-i}$.

Now we observe that the sum is just a special case of lemma 2, where *m* = *n*, and *k* = *n*, as follows:

□

$$\sum_{i=0}^{n} \binom{n}{n-i}\binom{n}{i} = \binom{n+n}{n}$$

Logic and Propositional Calculus

Propositional logic was eventually refined using symbolic logic. The 17th/18th century philosopher Gottfried Leibniz (an inventor of calculus) has been credited with being the founder of symbolic logic. Although his work was the first of its kind, it was unknown to the larger logical community. Consequently, many of the advances achieved by Leibniz were re-achieved by logicians like George Boole and Augustus De Morgan in the 19th century completely independent of Leibniz.

A ***proposition*** is a statement that is either true or false.
For example:
The earth is flat.
A tomato is a fruit.
The answer to the ultimate question of life, the universe, and everything is 42.

Basic operations

Let p and q be logical variables.

Basic operations are defined as:
Conjunction $p \wedge q$ (p and q)
(true if both p and q are true, otherwise false)

Disjunction $p \vee q$ (p or q)
(true if either p or q are true, otherwise false)

Negation $\neg p$ (not p)
(true if p is false (not true), otherwise false)

Truth tables

We can enumerate the values of logical expressions using a truth table.

For example:

p	q	¬q	p∧q	p∨q
T	T	F	T	T
T	F	T	F	T
F	T	F	F	T
F	F	T	F	F

Notation

We can denote a logical expression constructed from logical variables p,q, and logical operators $\wedge$,$\vee$, and $\neg$ (and, or, not) using the notation P(p,q).

We call this type of expression a ***logical proposition***.

For example: $\neg(p \vee q)$ (not (p or q)) is a logical proposition that depends on the values of p and q. We can use truth tables to determine truth values of a logical proposition.

p	q	$(p \vee q)$	$\neg(p \vee q)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

Definitions

A *tautology* is a logical expression that is always true for all values of its variables.

A *contradiction* is a logical expression that is always false (never true) for all values of its variables

q	**¬q**	**q∨¬q**	**q∧¬q**
T	**F**	**T**	**F**
F	**T**	**T**	**F**

Whether q is true or false, q ∨ ¬q is always true, and q ∧ ¬q is always false.

Logical Equivalence

Two propositions (using the same variables)
P(p,q) Q(p,q) are said to be *logically equivalent* or *equivalent* or *equal* if they have identical truth table values.

We notate equivalence:

$$P(p,q) \equiv Q(p,q)$$

There are a set of “laws” of logic that are very similar to the laws of set theory.

The laws of logic can be proved by using truth tables.

Table 4-1 Laws of the algebra of propositions

Idempotent laws:	(1a) $p \vee p \equiv p$	(1b) $p \wedge p \equiv p$
Associative laws:	(2a) $(p \vee q) \vee r \equiv p \vee (q \vee r)$	(2b) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
Commutative laws:	(3a) $p \vee q \equiv q \vee p$	(3b) $p \wedge q \equiv q \wedge p$
Distributive laws:	(4a) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	(4b) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
Identity laws:	(5a) $p \vee F \equiv p$ (6a) $p \vee T \equiv T$	(5b) $p \wedge T \equiv p$ (6b) $p \wedge F \equiv F$
Involution law:	(7) $\neg\neg p \equiv p$	
Complement laws:	(8a) $p \vee \neg p \equiv T$ (9a) $\neg T \equiv F$	(8b) $p \wedge \neg p \equiv T$ (9b) $\neg F \equiv T$
DeMorgan's laws:	(10a) $\neg(p \vee q) \equiv \neg p \wedge \neg q$	(10b) $\neg(p \wedge q) \equiv \neg p \vee \neg q$

We prove DeMorgan’s law with truth tables

p	**q**	**$\neg (p \vee q)$**
T	T	F
T	F	F
F	T	F
F	F	T

$\neg p$	**$\neg q$**	**$\neg p \wedge \neg q$**
F	F	F
F	T	F
T	F	F
T	T	T

We prove the distributive law with truth tables

p	q	r	p∨(q∧r)
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	F
F	F	T	F
F	F	F	F

p	q	r	(p∨q) ∧(p∨r)
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	F
F	F	T	F
F	F	F	F

Conditional Statements

A typical statement in mathematics is of the form "if p then q".

For example:

In all of these examples variables are assumed to be natural numbers.

if $a \leq b$ and $b \leq a$ then $a = b$

if $a-7 < 0$, then $a < 7$

if $2 \mid a$ then $2 \mid (a)(b)$

All of these statements are true if a and b are natural numbers.

In logic we use the symbol → to model this type of statement. However, using the symbol → in logic does not necessarily have a causal relationship between p and q.

"if p then q" is denoted $p \rightarrow q$, and pronounced either "if p then q" or "p implies q".

A truth table is used to define the outcomes when using the → logical operator.

p	q	p → q
T	T	T
T	F	F
F	T	T
F	F	T

This definition does not appear to make much sense, however, this is how implication is defined in logic.

if sugar is sweet then lemons are sour.
Is a true implication.
if sugar is sweet then the earth is flat.
Is a false implication.
if the earth is flat then sugar is sweet.
Is a true implication.
if the earth is flat then sugar is bitter.
Is a true implication

The truth table for implications can be summarized as:

1. An implication is true when the “if” part is false, or the “then” part is true.
2. An implication is false only when the “if” part is true, and the “then” part is false.

Note that $p \rightarrow q$ is logically equivalent to $\neg p \vee q$.

We can verify this with a truth table

p	q	$\neg p \vee q$
T	T	
T	F	
F	T	
F	F	

Biconditional Implications

A shorthand for the pair of statements

• **if** $a \le b$ and $b \le a$ **then** $a = b$
• **if** $a = b$ **then** $a \le b$ and $b \le a$
is:
$a = b$ **if and only if** $a \le b$ and $b \le a$

This can be notated as
$a = b \leftrightarrow (a \le b) \wedge (b \le a)$

An often used abbreviation for "if and only if" is "iff".

A truth table for the biconditional implication is:

p	q	p ↔ q
T	T	T
T	F	F
F	T	F
F	F	T

The truth table for biconditional implications can be summarized as:

1. A biconditional implication is true when both p and q are true, or both p and q are false.

Note that $p \leftrightarrow q$ is logically equivalent to
$(p \rightarrow q) \wedge (q \rightarrow p)$ as well as $(\neg p \vee q) \wedge (\neg q \vee p)$.

Suppose we have the proposition
$p \rightarrow q$
the *contrapositive*:
$\neg q \rightarrow \neg p$?
is logically equivalent as verified by the following truth table.

p	q	¬p	¬q	¬q → ¬p
T	T	F	F	T
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

The following example may help in understanding the contrapositive.

if 2 | a then 2 | (a)(b) is logically equivalent to
if 2 ∤ (a)(b) then 2 ∤ a.

Suppose we have the proposition
$p \rightarrow q$
the *<u>converse</u>*:
$q \rightarrow p$?
is not logically equivalent as verified by the following truth table.

p	**q**	**$q \rightarrow p$**	**$p \rightarrow q$**
T	T	T	T
T	F	T	F
F	T	F	T
F	F	T	T

The following example may help in understanding why the converse is not logically equivalent to the implication.

if 2 | a then 2 | (a)(b) is not logically equivalent to
if 2 | (a)(b) then 2 | a.

It should be obvious that an implication **and** its converse results in a biconditional implication.

that is:
$p \leftrightarrow q$ is logically equivalent to
$(p \rightarrow q) \land (q \rightarrow p)$