CISC-102
Fall 2017
Week 3

Principle of Mathematical Induction

A *proposition* is defined as a statement that is either true or false. We will at times make a declarative statement as a proposition and then proceed to prove that it is true. Alternately we may provide an example (called a *counterexample*) showing that the proposition is false.

Let P be a proposition defined on the positive integers $\mathbb{N}$; that is, P(n) is either true or false for each $n \in \mathbb{N}$. Suppose P has the following two properties:
(i) P(1) is true.
(ii) P(k+1) is true whenever P(k) is true.

Then by the principle of Mathematical Induction P is true for every positive integer $n \in \mathbb{N}$.

Mathematical induction is by far the most useful tool for proving results in computing.

Note: Step (i) may be replaced by any integer *b* and then the principle of mathematical induction would hold for all integers greater than or equal to *b*.

Example:

$$1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$$

for all natural numbers n .

We can verify that the equation holds for small values of n, say n = 1,2, 3. However this does not prove that the equation is true for all natural numbers n.

Let P(n) be the proposition that the equation above is correct for the natural number n. We will use mathematical induction to prove that P(n) is true for every $n \in \mathbb{N}$.

Base: $1 + 2^1 = 3 = 2^2 - 1$

Induction Hypothesis : P(k) is true, for some fixed value k, such that $k \geq 1$.

Induction Step: (Our goal is to prove that P(k+1) is true using the assumption that P(k) is true.)

$$1 + 2 + \cdots + 2^k + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1}$$

The equality holds because we replace

$$1 + 2 + \cdots + 2^k$$

with

$$2^{k+1} - 1$$

using the assumption that P(k) is true.

Now we do a bit of algebra and arithmetic.

$$= 2^{k+1} + 2^{k+1} - 1$$

because integer addition is commutative.

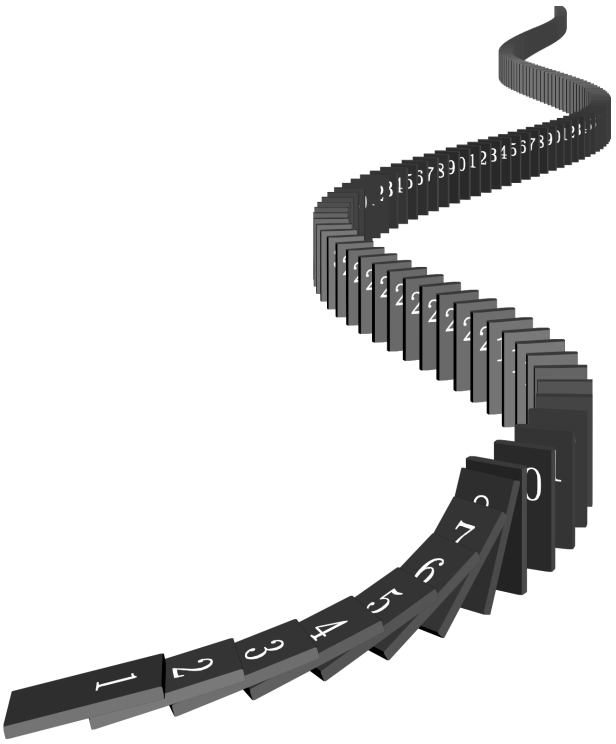
$$= 2^{k+2} - 1$$

If you follow the chain of equalities we have:

$$1 + 2^1 + \cdots + 2^{k+1} = 2^{k+2} - 1$$

and that P(k+1) is true.

Therefore by the principle of mathematical induction we conclude that P(n) is true for all natural numbers n. □



As an analogy think of an unending sequence of dominoes. You can be sure that all will fall if:

1. The first one falls. (P(1))
2. And if the k^{th} one falls it will knock over the $k+1^{st}$, that is, P(k) true implies that P(k+1) is also true.

Let P(n) be the proposition that the number of two element subsets of a set of n elements is given by the formula: $n(n\text{-}1)/2$.

We can show that P(n) is true for all natural numbers n by using mathematical induction.

Base: A set with 1 element has zero 2 element subsets, satisfying the equation $0 = 1(0)/2$.
Induction Hypothesis: P(k) is true for an arbitrary natural number $k \geq 1$.
Induction Step: Let S be a set with k+1 elements, and let $s \in S$. We can partition the set of two element subsets of S into the subsets that include the element s and those that don't. Let $S' = S \setminus \{s\}$ (that is S' contains all the elements in S except for s.)

Observe that the two element subsets of S' are exactly the two elements subsets of S that do not contain s. Furthermore S' has k elements so by the induction hypothesis S' has $k(k\text{-}1)/2$ two element subsets.

We can pair up *s* with every element of *S*' to get the two element subsets of *S* that do contain *s*, so that makes *k* additional two element subsets.

Summing these subsets yields the formula

$$\begin{aligned} k + \frac{k(k-1)}{2} &= \frac{2k + k(k-1)}{2} \\ &= \frac{2k + k^2 - k}{2} \\ &= \frac{k^2 + k}{2} \\ &= \frac{(k+1)(k)}{2} \end{aligned}$$

Therefore, we have shown that the proposition P(*k*) true implies that P(*k+1*) is true. So by the principle of mathematical induction we conclude that P(*n*) is true for all natural numbers n. □

You may be tempted to think that it is enough to just enumerate a few cases to convince yourself that a proposition is true.

Let P(n) be the proposition that $3n < 1000$ for all natural numbers n.

$3 \times 1 < 1000$
$3 \times 2 < 1000$
.
.
.
$3 \times 333 < 1000$

So P(n) must be true. (Obviously NOT!)

Here's another example where the first few cases lead to a false conclusion.

Let P(n) be the proposition that $n! < 2^n$ is true for all $n \in \mathbb{N}$.

Observe that:

$1! = 1 < 2$
$2! = 2 < 4$
$3! = 6 < 8$

However, if we check one additional case,

$4! = 24 > 16$.

In fact we can use induction to prove that

$n! \geq 2^n$, is true for all $n \in \mathbb{N}$, $n \geq 4$.

Students sometimes find proving results using inequalities (that is relations like $\leq$, $<$, $\geq$, $>$) hard to grasp. Don't worry if you don't get this the first time you read it. If you persist you should eventually understand this.

Theorem: $n! \geq 2^n$, for $n \in \mathbb{N}$, $n \geq 4$.

Proof: Let P(n) be the proposition $n! \geq 2^n$, for $n \geq 4$.

Base: P(4) is true because $4 \times 3 \times 2 \times 1 \geq 2^4$

Induction Hypothesis: P(k) is true for $k \geq 4$.

Induction Step: $(k + 1)\,! = k!\,(k+1)$

$\geq 2^k\,(k+1)$ (because P(k) is true)

$\geq 2^k\,2$ (because $k \geq 4$)

$\geq 2^{k+1}$

Therefore, we have shown that the proposition P(k) true implies that P($k+1$) is true. So by the principle of mathematical induction we have P(n) is true for all natural numbers $n \geq 4$. □

Example: The sum of the first n odd numbers is n^2.

Let's try it for some small values of n.
n = 1 (1 = 1^2), n = 2 (1+ 3 = 4 = 2^2),
n = 3 (1 + 3 + 5 = 9 = 3^2)
This is NOT A PROOF! These simply show that the propositions P(1), P(2) and P(3) are true.

Preliminaries: The k^{th} odd number can be written as $2k-1$.
e.g.[1] 1 = 2 ×1-1, 3 =2×2-1, 5=2×3-1 *etc*[2]. This fact will be useful for proving that the sum of the first n odd numbers is n^2.

At this point let's take a closer look at what is meant by an odd number and define it precisely.

Let n be a natural number.

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[1]exempli gratia are Latin words meaning "for example"

[2] et cetera are Latin words meaning "and so on"

Even Natural numbers

If 2 divides n, that is n/2 is a natural number then we say that n is even. For example 2/2 = 1 so 2 is even, 4/2 = 2 so 4 is even. Every even natural number can then be expressed as a multiple of 2. For example $2 \times 1 = 2$, and $2 \times 2 = 4$. So if k is a natural number $2k$ is even.

Odd Natural numbers

When we study integers and integer arithmetic we will be better equipped to formally define what is meant be an odd number. For now we can simply define an odd natural number as any natural number that is 1 less than an even number, that is $2k$-1.

Theorem: The proposition P(n), the sum of the first *n* odd numbers is n^2 for all natural numbers *n*.

Proof:

Base: P(1) $1 = 1^2$, so P(1), the base case is true.

Induction Hypothesis: Assume P(k) is true where k is any arbitrary integer greater than or equal to 1.

That is, $1 + 3 + 5 + ... + 2k-1 = k^2$.

Induction Step: Consider the sum of the first k+1 odd numbers.

$\underline{1 + 3 + 5 \ldots + 2k-1} + 2k+1$

$= \underline{k^2} + 2k+1$ (because P(k) is assumed true)

$= (k+1)(k+1)$ (factor)

$= (k+1)^2$

Therefore, we have shown that the proposition P(k) true implies that

P(k+1) is true. So by the principle of mathematical induction we conclude that P(n) is true for all natural numbers n. □

Some of you may have learned to resolve this type of sequence of equations as follows.

RHS	LHS
$1 + 3 + \ldots + 2K+1 =$	$(k+1)^2$
$k^2 + 2k + 1 =$	$(k+1)\ (k+1)$
$(k+1)(k+1) =$	$(k+1)(k+1)$

You can use this as a preliminary step but it is an abuse of notation. Once you have worked this preliminary step you can re-write the sequence by going down the right hand side, and then up the left hand side (omitting repeats).

Theorem: The proposition P(n), the sum of the first n even natural numbers is $n^2 + n$ for all natural numbers n.

Proof:

Base: P(1) $2 = 1^2 + 1$, so P(1), the base case is true.

Induction Hypothesis: Assume P(k) is true where k is any arbitrary integer greater than or equal to 1.

That is, $2 + 4 + 6 + ... + 2k = k^2 + k$.

Our goal in the Induction step is to show that

$2 + 4 + \ldots + 2k + 2(k + 1) = (k+1)^2 + k+1$.

Tip: $(k+1)^2 + k + 1 = k^2 + 2k + 1 + k + 1$

$= k^2 + 3k + 2$

Induction Step:

$2 + 4 \ldots + 2k + 2(k + 1) = k^2 + k + 2k + 2$

$= k^2 + 3k + 2$

Therefore, we have shown that the proposition P(k) true implies that P(k+1) is true. So by the principle of mathematical induction we conclude that P(n) is true for all natural numbers n. □