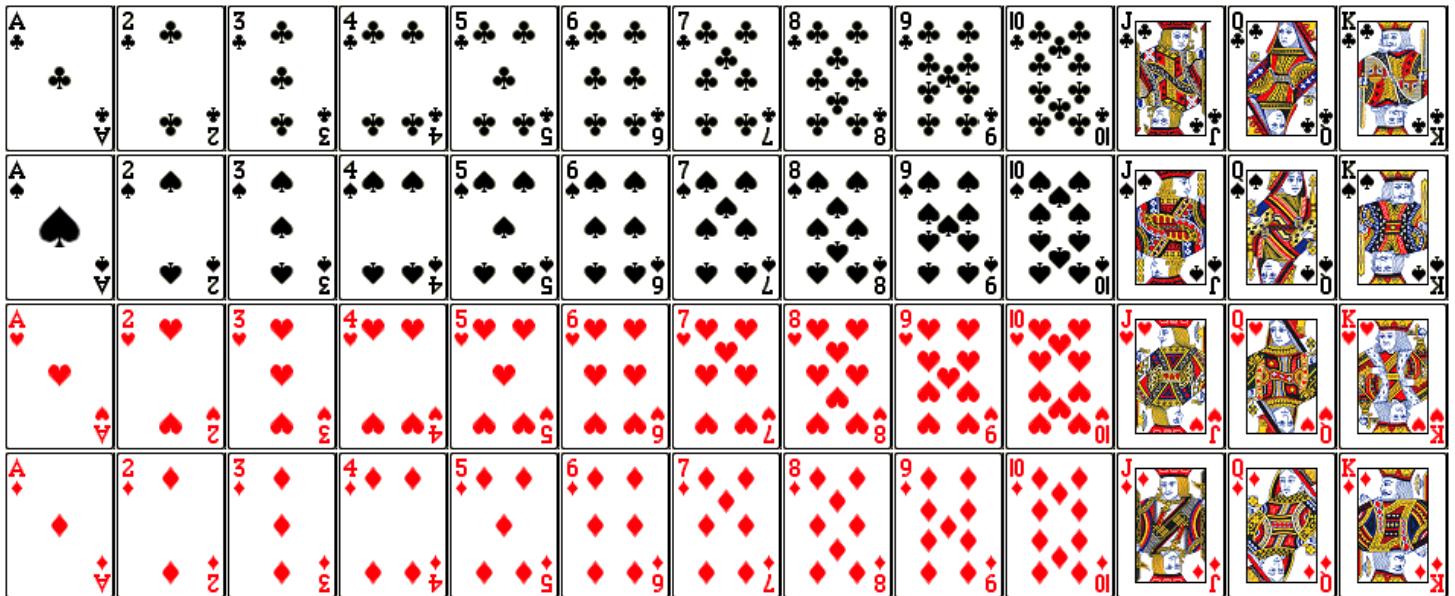


CISC-102
 Fall 2017
 Week 9

Playing cards.

Some of the following examples make use of the standard 52 deck of playing cards as shown below.



There are 4 suits (clubs, spades, hearts, diamonds) each consisting of 13 values (Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, King) for a total of 52 cards.

Counting Poker Hands.

Notation:

A card from a standard 52 card deck will be denoted using an ordered pair as follows:

(v,s) where v is an element of the set of 13 values:

$$\{A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, K, Q\}.$$

and s is an element of the set of 4 suits

$$\{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}.$$

is the suit of the card.

For this discussion a poker hand is a 5 card subset of the 52 card deck.

There are 2,598,960 different 5 card subsets from a 52 card deck.

How is this value obtained?

A hand with no straight or flush or 4,3, or 2 of a kind is called a no-pair. How do we count the number of 5 card no-pair hands.

A counting idea we can exploit is:

- Count the number of ways to get 5 different cards that are not a straight.
- Count the different suits for these 5 cards that do not make a flush.

So the number of 5 card no-pair hands is:

$$\left(\binom{13}{5} - \binom{10}{1} \right) \left(\binom{4}{1}^5 - \binom{4}{1} \right)$$

The probability of getting a no-pair when randomly selecting 5 cards is:

$$\frac{\left(\binom{13}{5} - \binom{10}{1} \right) \left(\binom{4}{1}^5 - \binom{4}{1} \right)}{\binom{52}{5}}$$

And this works out to be:

$$(1277)(1020)/2,598,960^1 = 1,302,540/2,598,960$$

or about 0.5.

You can find counting formulas for a variety of 5 card poker hands on this Wikipedia page:

https://en.wikipedia.org/wiki/Poker_probability

¹ I used Octave, an open source version of Matlab, to compute this value.

The Pigeon Hole Principle



If there are n pigeons, that all must sleep in a pigeon hole, and $n-1$ pigeon holes, then there is at least one pigeon hole where (at least) 2 pigeons sleep.

This should be obvious! Mathematicians give it a name because it is a useful counting tool.

Can we find two people living in the G.T.A. that have exactly the same number of strands of hair on their heads?

The answer is YES! And we can prove it using the pigeon hole principle.

The population of the G.T.A is more than 6 million.
Science tells us that nobody has more than 500,000 strands of hair on their heads.

To solve the problem using the pigeon hole principle we imagine 500,000 pigeon holes labelled from 1, ..., 500,000 and then imagine each resident of the G.T.A. entering the pigeon hole labelled with the number of strands of hair on their head. Since 6 million is greater than 500,000 we deduce that there will be at least one pigeon hole where two or more people have entered.

The Binomial Theorem

When we expand the expression:

$$(x + y)^3$$

we get:

$$(x+y)(x+y)(x+y) = x^3 + 3x^2y + 3xy^2 + y^3$$

this can also be written as follows:

$$(x + y)(x + y)(x + y) = \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3$$

We can reason that when we expand $(x + y)^3$, there is one way to choose a triple that is exclusively x's (with 0 y's), 3 ways to choose a triple that has 2 x's (and 1 y), and 3 ways to choose a triple that has 1 x (and 2 y's). Finally there is 1 way to choose a triple with no x (and 3 y's).

Binomial Theorem:

$$\begin{aligned}(x + y)^n &= \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \cdots + \binom{n}{n} x^0 y^n \\ &= \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k\end{aligned}$$

For all natural numbers n .

Proof: In the expansion of the product:

$$(x + y) (x + y) \cdots (x+y),$$

there $\binom{n}{k}$ ways to choose an n -tuple with $n-k$ x 's and k y 's). \square

A special case of the binomial theorem should look familiar.

$$\begin{aligned}(1 + 1)^n &= \binom{n}{0} 1^n 1^0 + \binom{n}{1} 1^{n-1} 1 + \binom{n}{2} 1^{n-2} 1^2 + \dots + \binom{n}{n} 1^0 1^n \\ &= \sum_{k=0}^n \binom{n}{k}\end{aligned}$$

This is just the sum of the sizes of all subsets of a set of size n .