CISC-102

Fall 2019

Week 10

When we expand the expression:
$(x + y)^3$

we get:
$(x+y)(x+y)(x+y) = x^3 + 3x^2y + 3xy^2 + y^3$

this can also be written as follows:

$$(x+y)(x+y)(x+y) = \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3$$

We can reason that when we expand $(x + y)^3$, there is one way to choose a triple that is exclusively x's (with 0 y's), 3 ways to choose a triple that has 2 x's (and 1 y) , and 3 ways to choose a triple that has 1 x (and 2 y's). Finally there is 1 way to choose a triple with no x (and 3 y's).

Binomial Theorem:

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n}x^0 y^n$$
$$= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

For all natural numbers *n*.

Proof: In the expansion of the product:

$$(x + y)\,(x + y) \cdots (x+y),$$

there $\binom{n}{k}$ ways to choose an *n*-tuple with n-*k* *x*'s and (*k* *y*'s). $\square$

A special case of the binomial theorem should look familiar.

$$
\begin{aligned}
(1+1)^n &= \binom{n}{0}1^n1^0 + \binom{n}{1}1^{n-1}1 + \binom{n}{2}1^{n-2}1^2 + \cdots + \binom{n}{n}1^01^n \\
&= \sum_{k=0}^{n}\binom{n}{k}
\end{aligned}
$$

This is just the sum the sizes of
all subsets of a set of size *n*.

Using counting to prove theorems.

Counting arguments can be useful tool for proving theorems. In each case there is also an algebraic way of proving the result. However, there is an inherent beauty in the elegant simplicity of some of these counting arguments so it's well worth looking at some examples. These proofs lack the formality of algebraic proofs. The lack of formality may make these arguments harder to grasp for some, and easier to understand for others.

The proofs we see will be to prove the validity of equations. We will count the left and right hand side of each equation and show that they count the same thing.

Binomial Coefficients

We prove identities involving binomial coefficients using counting arguments.

Theorem:

$$\binom{n}{k} = \binom{n}{n-k}$$

Proof: On the left we have the quantity $\binom{n}{k}$ which represents the number of ways to select a *k* element subset from an *n* element set, *S*. Using the analogy of selecting balls from a bag, we see that we also implicitly select the complementary subset that stays in the bag, and the number of ways to do this is as given on the right hand side of the equation is $\binom{n}{n-k}$. $\square$

Theorem A:

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

Proof: On the left the quantity $\binom{n+1}{k}$ represents the number of ways to select a k element subset from an $n+1$ element set. To see what the right hand side counts we suppose that there is a "favourite" or "distinguished" element of the set, call it x.
The number of ways to select a k element subset from $n+1$ distinct objects that is guaranteed to *include* x is to pull x out and then choose the remaining $k-1$ elements in $\binom{n}{k-1}$ ways. On the other hand the number of ways to select a k element subset from $n+1$ distinct objects that is guaranteed to *exclude* x is to pull x out and then choose all k elements in $\binom{n}{k}$ ways.

Therefore the left and right hand side both count the same thing thus justifying the equation. □

And here's an alternate algebraic proof.

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$$

Proof:

$$\begin{aligned}
\binom{n}{k-1} + \binom{n}{k} &= \frac{n!}{(n-k+1)!(k-1)!} + \frac{n!}{(n-k)!(k)!} \\
&= \frac{n!k + n!(n-k+1)}{(n+1-k)!k!} \\
&= \frac{n!(k+n-k+1)}{(n+1-k)!k!} \\
&= \frac{n!(n+1)}{(n+1-k)!k!} \\
&= \frac{(n+1)!}{(n+1-k)!k!} \\
&= \binom{n+1}{k}
\end{aligned}$$

Theorem:

$$\sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$$

Proof: On the left the sum counts all the subsets of a set of size n. We already know that the number of subsets of a set of size n, is 2^n.

Therefore the left and right hand side both count the same thing thus justifying the equation. $\square$

Pascal's Triangle

An easy way to calculate a table of binomial coefficients was recognized centuries ago by mathematicians in India, China, Iran and Europe.
In the west the technique is named after the French mathematician Blaise Pascal (1623-1662). In the example below each row represents the binomial coefficients as used in the binomial theorem.

$$\begin{array}{ccccccccccccc}
 & & & & & & \binom{0}{0} & & & & & & \\
 & & & & & \binom{1}{0} & & \binom{1}{1} & & & & & \\
 & & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & & \\
 & & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & & \\
 & & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & & \\
 & \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} & \\
\binom{6}{0} & & \binom{6}{1} & & \binom{6}{2} & & \binom{6}{3} & & \binom{6}{4} & & \binom{6}{5} & & \binom{6}{6}
\end{array}$$

To obtain the entries by hand in a simple way we can use the identity:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

$$\begin{array}{ccccccccccccc}
 & & & & & & \binom{0}{0} & & & & & & \\
 & & & & & \binom{1}{0} & & \binom{1}{1} & & & & & \\
 & & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & & \\
 & & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & & \\
 & & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & & \\
 & \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} & \\
\binom{6}{0} & & \binom{6}{1} & & \binom{6}{2} & & \binom{6}{3} & & \binom{6}{4} & & \binom{6}{5} & & \binom{6}{6}
\end{array}$$

$$\begin{array}{ccccccccccccccccc}
 & & & & & & & & 1 & & & & & & & & \\
 & & & & & & & 1 & & 1 & & & & & & & \\
 & & & & & & 1 & & 2 & & 1 & & & & & & \\
 & & & & & 1 & & 3 & & 3 & & 1 & & & & & \\
 & & & & 1 & & 4 & & 6 & & 4 & & 1 & & & & \\
 & & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & & \\
 & & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & & \\
 & 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 & \\
1 & & 8 & & 28 & & 56 & & 70 & & 56 & & 28 & & 8 & & 1
\end{array}$$

Consider the sum of elements in a row of Pascal's triangle. If we label the top row 0, then it appears that row i sums to the value 2^i. Can you explain why this is the case?

$$
\begin{array}{c}
\binom{0}{0} \\
\binom{1}{0} \quad \binom{1}{1} \\
\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2} \\
\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3} \\
\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4} \\
\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5} \\
\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}
\end{array}
$$

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1
\end{array}
$$

Now let's compute the sum of squares of the entries of each row in Pascal's triangle.

$1^2 = 1$
$1^2 + 1^2 = 2$
$1^2 + 2^2 + 1^2 = 6$
$1^2 + 3^2 + 3^2 + 1^2 = 20$
$1^2 + 4^2 + 6^2 + 4^2 + 1^2 = 70$

These sums all appear in the middle row of Pascal's triangle.

$$
\begin{array}{ccccccccccccccccc}
&&&&&&&&1&&&&&&&& \\
&&&&&&&1&&1&&&&&&& \\
&&&&&&1&&2&&1&&&&&& \\
&&&&&1&&3&&3&&1&&&&& \\
&&&&1&&4&&6&&4&&1&&&& \\
&&&1&&5&&10&&10&&5&&1&&& \\
&&1&&6&&15&&20&&15&&6&&1&& \\
&1&&7&&21&&35&&35&&21&&7&&1& \\
1&&8&&28&&56&&70&&56&&28&&8&&1
\end{array}
$$

Which leads us to conjecture that:

$$\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}$$

Before proving the theorem there are two preliminary lemmas.

Lemma 1:

$$\binom{n}{k}\binom{n}{n-k} = \binom{n}{k}^2$$

For all non-negative integers *n*,*k*, *n* > *k*.

Proof: Since we already showed that $\binom{n}{k} = \binom{n}{n-k}$ this should be obvious. □

$$\sum_{i=0}^{k} \binom{m}{k-i}\binom{n}{i} = \binom{m+n}{k}$$

Lemma 2:

For all non-negative integers *m,n,k* such that $n \geq m \geq k$.

Proof: We use a counting argument. The right hand side can be viewed as the number of subsets of size *k* chosen from the union of two disjoint sets, *S* of size *m*, and *T* of size *n*. On the left we sum the choices where all *k* are from *S*, then *k-1* from *S* and 1 from *T* and so on up to all *k* chosen from set *T*. □

For example: Suppose
S = {a,b} with |S| = *m* = 2, and
T = {c,d,e} with |T| = *n* = 3 and
k = 2. So the sum on the right would be:

$$\sum_{i=0}^{2} \binom{2}{2-i}\binom{3}{i} = \binom{2}{2}\binom{3}{0} + \binom{2}{1}\binom{3}{1} + \binom{2}{0}\binom{3}{2}$$

Theorem:

$$\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}$$

for all natural numbers n ≥ 1.

Proof: Using lemma 1 we can write $\binom{n}{i}^2 = \binom{n}{i}\binom{n}{n-i}$.

Now we observe that the sum is just a special case of lemma 2, where *m* = *n*, and *k* = *n*, as follows:

$\square$

$$\sum_{i=0}^{n} \binom{n}{n-i}\binom{n}{i} = \binom{n+n}{n}$$

Logic and Propositional Calculus (SN Chapter 4.)

Propositional logic was eventually refined using symbolic logic. The 17th/18th century philosopher Gottfried Leibniz (an inventor of calculus) has been credited with being the founder of symbolic logic. Although his work was the first of its kind, it was unknown to the larger logical community. Consequently, many of the advances achieved by Leibniz were re-achieved by logicians like George Boole and Augustus De Morgan in the 19th century completely independent of Leibniz.

A *proposition* is a statement that is either true or false.
For example:
The earth is flat.
A tomato is a fruit.
The answer to the ultimate question of life, the universe, and everything is 42.[1]

[1] Quoted from: Douglas Adams, “The Hitchhiker’s Guide to the Galaxy” (1979).

Basic operations

Let p and q be logical variables.

Basic operations are defined as:
Conjunction $p \wedge q$ (p and q)
(true if both p and q are true, otherwise false)

Disjunction $p \vee q$ (p or q)
(true if either p or q are true, otherwise false)

Negation $\neg p$ (not p)
(true if p is false (not true), otherwise false)

Truth tables

We can enumerate the values of logical expressions using a truth table.

For example:

p	q	¬q	p∧q	p∨q
T	T	F	T	T
T	F	T	F	T
F	T	F	F	T
F	F	T	F	F

Notation

We can denote a logical expression constructed from logical variables p,q, and logical operators $\wedge$,$\vee$, and $\neg$ (and, or, not) using the notation P(p,q).

We call this type of expression a ***logical proposition***.

For example: $\neg(p \vee q)$ (not (p or q)) is a logical proposition that depends on the values of p and q. We can use truth tables to determine truth values of a logical proposition.

p	q	$(p \vee q)$	$\neg(p \vee q)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

Definitions

A *tautology* is a logical expression that is always true for all values of its variables.
A *contradiction* is a logical expression that is always false (never true) for all values of its variables

q	**¬q**	**q∨¬q**	**q∧¬q**
T	**F**	**T**	**F**
F	**T**	**T**	**F**

Whether q is true or false, q ∨ ¬q is always true,
and q ∧ ¬q is always false.

Logical Equivalence
Two propositions (using the same variables) P(p,q) Q(p,q) are said to be *logically equivalent* or *equivalent* or *equal* if they have identical truth table values.
We notate equivalence:

$$P(p,q) \equiv Q(p,q)$$

There are a set of “laws” of logic that are very similar to the laws of set theory.

The laws of logic can be proved by using truth tables.

Table 4-1 Laws of the algebra of propositions

Idempotent laws:	(1a) $p \vee p \equiv p$	(1b) $p \wedge p \equiv p$
Associative laws:	(2a) $(p \vee q) \vee r \equiv p \vee (q \vee r)$	(2b) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
Commutative laws:	(3a) $p \vee q \equiv q \vee p$	(3b) $p \wedge q \equiv q \wedge p$
Distributive laws:	(4a) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	(4b) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
Identity laws:	(5a) $p \vee F \equiv p$ (6a) $p \vee T \equiv T$	(5b) $p \wedge T \equiv p$ (6b) $p \wedge F \equiv F$
Involution law:	(7) $\neg\neg p \equiv p$	
Complement laws:	(8a) $p \vee \neg p \equiv T$ (9a) $\neg T \equiv F$	(8b) $p \wedge \neg p \equiv T$ (9b) $\neg F \equiv T$
DeMorgan's laws:	(10a) $\neg(p \vee q) \equiv \neg p \wedge \neg q$	(10b) $\neg(p \wedge q) \equiv \neg p \vee \neg q$

We prove DeMorgan's law with truth tables

p	**q**	**$\neg(p \vee q)$**
T	T	F
T	F	F
F	T	F
F	F	T

$\neg p$	**$\neg q$**	**$\neg p \wedge \neg q$**
F	F	F
F	T	F
T	F	F
T	T	T

We prove the distributive law with truth tables

p	**q**	**r**	**$p \vee (q \wedge r)$**
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	F
F	F	T	F
F	F	F	F

p	**q**	**r**	**$(p \vee q) \wedge (p \vee r)$**
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	F
F	F	T	F
F	F	F	F

Conditional Statements

A typical statement in mathematics is of the form
"if p then q".

For example:

In all of these examples variables are assumed to be natural numbers.

if $a \leq b$ and $b \leq a$ then $a = b$

if $a - 7 < 0$, then $a < 7$

if $2 \mid a$ then $2 \mid (a)(b)$

All of these statements are true if a and b are natural numbers.

In logic we use the symbol $\rightarrow$ to model this type of statement. However, using the symbol $\rightarrow$ in logic does not necessarily have a causal relationship between p and q.

"if p then q" is denoted $p \rightarrow q$, and pronounced either "if p then q" or "p implies q".

A truth table is used to define the outcomes when using the → logical operator.

p	q	p → q
T	T	T
T	F	F
F	T	T
F	F	T

This definition does not appear to make much sense, however, this is how implication is defined in logic.

if sugar is sweet then lemons are sour.
Is a true implication.
if sugar is sweet then the earth is flat.
Is a false implication.
if the earth is flat then sugar is sweet.
Is a true implication.
if the earth is flat then sugar is bitter.
Is a true implication

The truth table for implications can be summarized as:

1. An implication is true when the “if” part is false, or the “then” part is true.
2. An implication is false only when the “if” part is true, and the “then” part is false.

Note that $p \rightarrow q \equiv \neg p \vee q$.

We can verify this with a truth table

p	q	$\neg p \vee q$
T	T	
T	F	
F	T	
F	F	

Biconditional Implications

A shorthand for the pair of statements

- **if** $a \le b$ and $b \le a$ **then** $a = b$
- **if** $a = b$ **then** $a \le b$ and $b \le a$

is:
$a = b$ **if and only if** $a \le b$ and $b \le a$

This can be notated as
$a = b \leftrightarrow (a \le b) \wedge (b \le a)$

An often used abbreviation for “if and only if” is “iff”.

A truth table for the biconditional implication is:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

The truth table for biconditional implications can be summarized as:

1. A biconditional implication is true when both p and q are true, or both p and q are false.

Note that:

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

as well as:

$$p \leftrightarrow q \equiv (\neg p \vee q) \wedge (\neg q \vee p).$$

The following truth table verifies the logical equivalence
$p \leftrightarrow q \equiv (\neg p \vee q) \wedge (\neg q \vee p)$

p	q	$p \leftrightarrow q$	$\neg p \vee q$	$\neg q \vee p$	$(\neg p \vee q) \wedge (\neg q \vee p)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

Suppose we have the proposition
$p \rightarrow q$
the *contrapositive*:
$\neg q \rightarrow \neg p$?
is logically equivalent as verified by the following truth table.

p	**q**	**¬p**	**¬q**	**¬q → ¬p**
T	T	F	F	T
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

The following example may help in understanding the contrapositive.

if $2 \mid a$ then $2 \mid (a)(b)$ is logically equivalent to
if $2 \nmid (a)(b)$ then $2 \nmid a$.

Suppose we have the proposition
$p \rightarrow q$
the *<u>converse</u>*:
$q \rightarrow p$?
is not logically equivalent as verified by the following truth table.

p	**q**	**$q \rightarrow p$**	**$p \rightarrow q$**
T	T	T	T
T	F	T	F
F	T	F	T
F	F	T	T

The following example may help in understanding why the converse is not logically equivalent to the implication.

if $2 \mid a$ then $2 \mid (a)(b)$ is not logically equivalent to
if $2 \mid (a)(b)$ then $2 \mid a$.

It should be obvious that an implication **and** its converse results in a biconditional implication.

that is:
$p \leftrightarrow q$ is logically equivalent to
$(p \rightarrow q) \wedge (q \rightarrow p)$
or $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$.

Logical Consequence and Arguments

Consider the expression:

p is true and p implies q is true , as a consequence we can deduce that q must be true.

This is a logical argument, and can be written symbolically as,

$$p, p \rightarrow q \vdash q$$

where: $p, p \rightarrow q$ is called a sequence of *premises,* and q is called the *conclusion*.
The symbol $\vdash$ denotes a logical consequence.

A sequence of premises whose logical consequence leads to a conclusion is called an *argument*.

Valid Argument

We can now formally define what is meant by a valid argument.

> **The argument $P_1, P_2, P_3, \ldots, P_n \vdash Q$ is valid if and only if $P_1 \wedge P_2 \wedge P_3 \wedge \ldots \wedge P_n \rightarrow Q$ is a tautology.**

Example: Consider the argument

$$p \rightarrow q,\ q \rightarrow r,\ \vdash p \rightarrow r$$

We can see if this argument is valid by using truth tables to show that the proposition:

$$(p \rightarrow q) \wedge (q \rightarrow r) \rightarrow (p \rightarrow r)$$

a tautology, that is, the proposition is true for all T/F values of p,q,r.

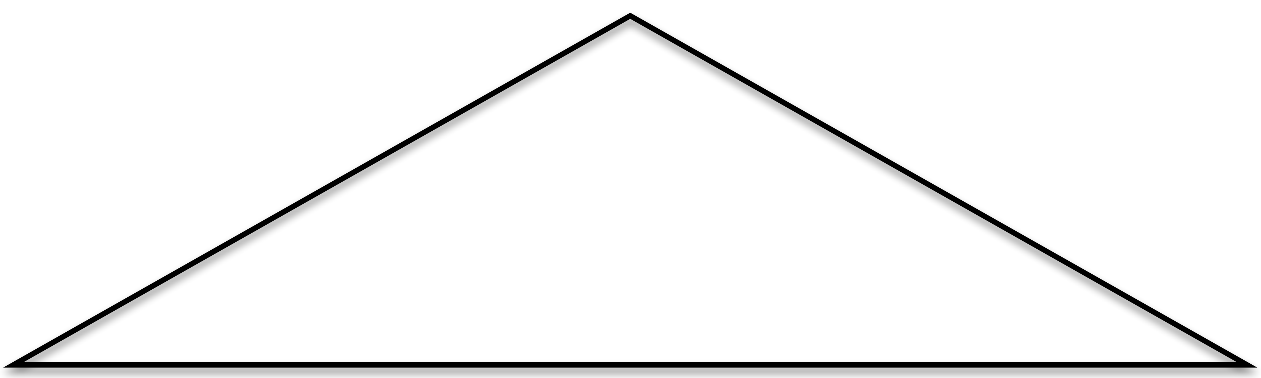
p	q	r	$(p \rightarrow q) \wedge (q \rightarrow r)$	$(p \rightarrow r)$	$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$
T	T	T	T	T	T
T	T	F	F	F	T
T	F	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	T	F	F	T	T
F	F	T	T	T	T
F	F	F	T	T	T

Consider the following argument:

If two sides of a triangle are equal **then**
the opposite angles are equal
T is a triangle with two sides that are not equal

The opposite angles of T are not equal

(With this notation the horizontal line separates a sequence of propositions from a conclusion.)



Let p be the proposition
"two sides of a triangle are equal"
and let q be the proposition
"the opposite angles are equal"

We can re-write the argument in symbols as:

$$p \to q, \neg p \vdash \neg q$$

and as the expression:

$$[(p \to q) \wedge \neg p] \to \neg q$$

We can check whether this is a valid argument by using a truth table to determine whether the expression is a tautology.

p	**q**	$[(p \rightarrow q) \wedge \neg p] \rightarrow \neg q$
T	T	
T	F	
F	T	
F	F	

Let's look at another logical argument that can be expressed as:

$[(p \rightarrow q) \wedge \neg p] \rightarrow \neg q$

If $2 \mid a$ **then** $2 \mid ab$
$2 \nmid a$

$2 \nmid ab$

(With this notation the horizontal line separates a sequence of propositions from a conclusion.)

Here we have

p the proposition: $2 \mid a$
q the proposition: $2 \mid ab$

I can show that the argument is invalid with an example:

Let a = 3 and b = 2. Clearly, $2 \nmid a$ and $2 \mid ab$

In the geometry argument we can't find a counter example. The reasoning is flawed, but we can obtain a correct version of the argument by noticing that:

If two sides of a triangle are equal **then**
the opposite angles are equal

Is a valid geometric fact. We also have the fact that:

If two angles of a triangle are equal **then**
the opposite sides are equal

(NOTE: if 2 | ab then 2 |a is not necessarily true)

The following is a valid argument.

If two angles of a triangle are equal **then**
then opposite sides are equal
T is a triangle with two sides that are not equal

The opposite angles of T are not equal

That is:

$[(p \to q) \wedge \neg q] \to \neg p$ is a valid argument and can be verified to be a tautology using truth tables

p	**q**	$(p \rightarrow q)$	$(p \rightarrow q) \wedge \neg q$	$[(p \rightarrow q) \wedge \neg q] \rightarrow \neg p$
T	T	T	F	T
T	F	F	F	T
F	T	T	F	T
F	F	T	T	T

Propositional Functions

Let $P(x)$ be a propositional function that is either true or false for each x in A.

That is, the domain of $P(x)$ is a set A, and the range is {true, false}. NOTE: Sometimes propositional function are called *predicates*.

Observe that the set A can be partitioned into two subsets:

- Elements with an image that is true.
- Elements with an image that is false.

In particular we may define the *truth set* of $P(x)$ as:
$T_P = \{ x : x \text{ in } A, P(x) \text{ is true}\}$

Examples: Consider the following propositional functions defined on the positive integers.

$P(x)$: $x + 2 > 7$; $T_P = \{x : x > 5\}$
$P(x)$: $x + 5 < 3$; $T_P = \varnothing$
$P(x)$: $x + 5 > 1$; $T_P = \mathbb{N}$

Quantifiers
There are two widely used logical quantifiers

Definition:
Universal Quantifier: $\forall$ (for all)

Let P(x) be a propositional function. A quantified proposition using the propositional function can be stated as:

$(\forall x \in A)\ P(x)$ (for all x in A P(x) is true)

$Tp = \{x : x \in A, P(x)\} = A$

Or if the elements of A can be enumerated as:

$A = \{x_1, x_2, x_3, \ldots\}$

We would have:

$P(x_1) \wedge P(x_2) \wedge P(x_3) \wedge \ldots$ is true.

Definition:
Existential Quantifier: ∃(there exists)

Let P(x) be a propositional function. A quantified proposition using the propositional function can be stated as:

$(\exists x \in A)\ P(x)$ (There exists an x in A *s.t.* P(x) is true)

$T_P = \{x : x \in A, P(x)\} \neq \varnothing$

Or if the elements of A can be enumerated as:

$A = \{x_1, x_2, x_3, ...\}$

We would have:

$P(x_1) \vee P(x_2) \vee P(x_3) \vee ...$ is true.

Quantifiers

Statement	True when:	False when:
$(\forall x \in A)\ P(x)$	$P(x)$ is true for every $x \in A$.	$P(x)$ is false for one or more $x \in A$.
$(\exists x \in A)\ P(x)$	$P(x)$ is true for one or more $x \in A$.	$P(x)$ is false for every $x \in A$.

Propositional functions with more than one variable

Consider the following illustrative example:

Let p(x,y) be the proposition that "x+y = 10" where the ordered pair $(x,y) \in \{1, 2, ..., 9\} \times \{1, 2, ..., 9\}$.

Consider the following quantified statements:

1. $\forall x \; \exists y \;\; p(x,y)$
2. $\exists y \; \forall x \; p(x,y)$

1. Says: "for every x there exists a y such that x + y = 10"
2. Says: "there exists a y such that for every x, x+y = 10"

Statement 1. is true, and statement 2, is false by inspection. This simply illustrates that the concepts that we have seen can be extended to more that one variable.