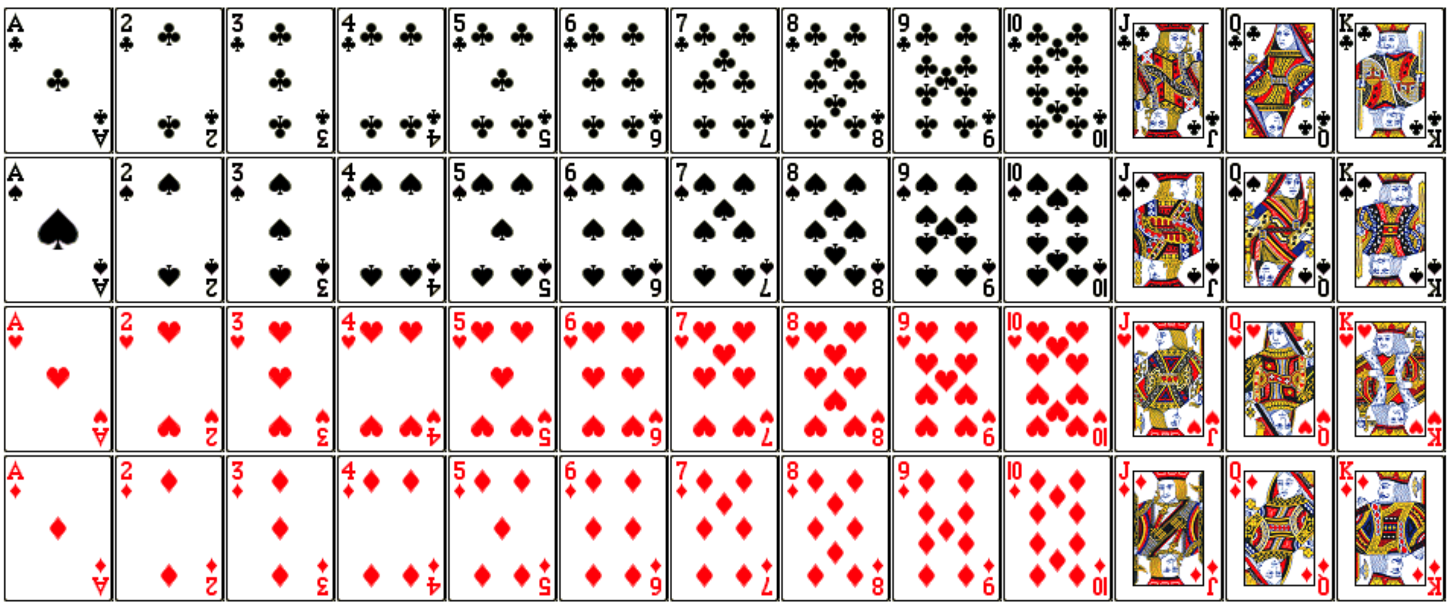
CISC-102
Fall 2019
Week 9

Techniques of Counting (Chapter 5 of SN)

Playing cards.



Some of the following examples make use of the standard 52 deck of playing cards as shown below.

There are 4 suits (clubs, spades, hearts, diamonds) each consisting of 13 values (Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, King) for a total of 52 cards.

Permutations

A common paradigm for counting is to imagine selecting labeled balls from a bag, so that no two balls are alike.

A permutation of objects is represented by a record of the order in which balls are pulled out of the bag.

Example: How many ways are there to select 5 different coloured balls from a bag?

$$5 \times 4 \times 3 \times 2 \times 1 = 5!$$

We can relate this to the product rule by thinking of the full bag as the set B_5, the bag with 4 balls as the set B_4, the bag with 3 balls B_3, the bag with 2 balls B_2, and with 1 ball B_1. Thus pulling balls from a bag can be viewed as a combination of the events (sets of outcomes) B_1, B_2, B_3, B_4, B_5. And the number of ways the combination of these events can occur as:

$$|B_1| \times |B_2| \times |B_3| \times |B_4| \times |B_5| = 5 \times 4 \times 3 \times 2 \times 1 = 5!$$

Example: How many different ways are there to shuffle a deck of cards?

We can number the cards in a deck from 1 to 52 where 1 is the card on top and 52 is the card on the bottom. So shuffling a deck of cards is equivalent to assigning a unique number from 1 ... 52 to each of the cards.

Observe that there is a bijection between the number of ways to draw balls from a bag, and the number of ways to select positions in a shuffled deck of cards. There are 52 positions to select as represented by the the following expression.

$$52 \times 51 \times 50 \ldots \times 1 = 52!$$

A permutation of the elements of a set is in essence assigning an ordering to a set.

Permutation rule
There are n! ways to permute n elements.

Example
Larry has 6 distinguishable pairs of socks. Each day Monday to Saturday he wears a different pair of socks. On Sunday he washes the socks (and goes sock-less). In how many different ways can Larry wear a week's worth of socks?

Permutation of a Subset

Suppose we want to count the number of ways of selecting 2 coloured balls from a total of 5 coloured balls.

$$5 \times 4 = 5!/3!$$

Suppose we want to count the number of ways to make an ordered selection of just 5 of the 52 cards.

$$52 \times 51 \times 50 \times 49 \times 48 = 52!/47!$$

different ways.

NOTATION:

$$P(n,k) = n!/(n-k)!$$

represents the number of permutations of k elements chosen from a collection of n elements.

Using our Poker hand analogy, a 5 card poker hand drawn from a 52 card deck one at a time, where <u>order</u> is taken into account has:

$$52 \times 51 \times 50 \times 49 \times 48 = 52!/(52-5)! = 52!/47!$$

different ways of occurring.

Permutations with Repetition

How many different ways can we order the letters:

BABY?

You may be temped to say 4! = 24 different ways, (that is select 4 balls labelled B A B Y from a bag) but upon inspection we see that there are only 12 distinguishable ways to order the letters.

The list of all 24 permutations that you see come in pairs.

BABY BABY	BYAB BYAB	AYBB AYBB
BAYB BAYB	BYBA BYBA	YBBA YBBA
BBAY BBAY	ABYB ABYB	YBAB YBAB
BBYA BBYA	ABBY ABBY	YABB YABB

I used colour to distinguish between the two B's in BABY. However, in reality the two B's are not distinguishable, and the list really should look like:

BABY BABY	BYAB BYAB	AYBB AYBB
BAYB BAYB	BYBA BYBA	YBBA YBBA
BBAY BBAY	ABYB ABYB	YBAB YBAB
BBYA BBYA	ABBY ABBY	YABB YABB

The correct way to count this is 4!/2! because two of the letters in B A B Y are identical.

How many ways are there to order the letters CCCB?

BCCC	CCBC
CBCC	CCCB

There are $4!/3! = 4$ ways

How many ways are there to order the letters BBCC?

BBCC	CBBC
BCBC	CBCB
BCCB	CCBB

There are $4!/2!2! = 6$ ways

Example: How many ways are there to pick ten coloured balls from a bag where each colour appears twice, so that two balls of the same colour are indistinguishable?

$$\frac{10!}{2!2!2!2!2!} = \frac{10!}{(2!)^5}$$

The counting formula is: The number of permutations of n objects consisting of $n_1, n_2, n_3, \ldots, n_r$ that are alike is:

$$\frac{n!}{n_1!n_2!\ldots n_r!}$$

Combinations

Suppose on the other hand that we want to count the number of different 5 card poker hands. We are interested in the number of ways of selecting 5 from 52 <u>without regard to the way that they are ordered</u>. We can solve this counting problem by answering the following questions.

(1) How many ways are there to shuffle a 5 card deck?

Answer: 5!

(2) How many ways are there to make an ordered selection of 5 of the 52 cards?

Answer: 52!/47!

(3) How do we put these two answers together to count the number of ways to make an un-ordered selection of 5 of the 52 cards?

Answer: We divide the answer to (2) by the answer to (1), yielding: 52!/(47!5!).

Combinations

We can use the balls in a bag analogy to count combinations. In this case we count the number of different ways to select distinct balls without ordering. The counting technique is a 2 step process.

1. Count the number of ways to select k balls from a bag of n balls with ordering.
2. Divide by the number of ways to order the k selected balls.

The outcome of this process yields the formula:

$$\frac{n!}{(n-k)!k!}$$

We have seen this expression before and the accompanying shorthand, that is:

$$\frac{n!}{(n-k)!k!} = \binom{n}{k}$$

NOTATION: $C(n,k) = P(n,k)/k! = \binom{n}{k}$

Counting Poker Hands.

Notation:

A card from a standard 52 card deck will be denoted using an ordered pair as follows:

(v,s) where v is an element of the set of 13 values:

$$\{A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, K, Q\}.$$

and s is an element of the set of 4 suits:

$$\{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}.$$

For this discussion a poker hand is a 5 card subset of the 52 card deck.

There are 2,598,960 different 5 card subsets from a 52 card deck.

How is this value obtained?

The most valuable poker hand is a royal flush, that is a 5 card subset that consists of the values 10,J,Q,K,A all in the same suit.

For example:

{(10,♧), (J,♧), (Q,♧), (K, ♧), (A,♧)}

is one example of a royal flush.

How many royal flushes are there?

There are exactly 4 different royal flushes. The “odds” of obtaining a royal flush is expressed as a ratio of all non royal flush 5 card poker hands versus royal flushes.

This ratio is:

(2,598,960-4):4

And simplifies to 649,739 : 1.

The next highest hand is a straight flush. That is a hand of 5 consecutive values (where A = 1 or A =14 as appropriate) all of the same suit. Normally the designation straight flush excludes the royal flushes.

There are a total of

$$\binom{10}{1}\binom{4}{1} - \binom{4}{1}$$

straight flushes.

A four of a kind consists of 4 cards of the same value plus one additional card.

Let's look at two equivalent ways of counting the number of 4 of kind hands.

1. Count the number of ways to select the value of the four of a kind (13) and then the number of ways to choose the 5th card (48), and multiply.
2. Count the number of ways to select the value of the four of a kind (13) and then number of ways to select the suit of the value of the 5th card (12) and the number of ways to select the suit of the 5th card (4), and multiply.

The odds of getting a four of a kind is 4,164 : 1

To see why we compute the product:

13(48) = 624

So there are 2,598,960 - 624 = 2,598,336 ways to get a "non four of a kind" vs. 624 ways to get a 4 of a kind, giving:

2,598,336:624 odds

which simplifies to:

4,164 : 1

A full house consists of 3 cards of the same value plus 2 cards of the same value?

For example: $\{7\clubsuit, 7\diamondsuit, 7\spadesuit, 3\heartsuit, 3\spadesuit\}$ is a full house.

There are:

$$\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}$$

ways to get a full house.

A poker hand is called 3 of a kind, when 3 cards have the same value, and the other two can be any two of the remaining values.

For example: $\{7\clubsuit, 7\diamondsuit, 7\spadesuit, 2\heartsuit, 3\spadesuit\}$ makes 3 of a kind.

How many different 3 of a kind hands are there?

A poker hand is called two pair if it consists of two distinct pairs of the same value and a 5th card with value different from the first two.

An <u>incorrect</u> way to count this is:

There are

$$\binom{13}{1}\binom{4}{2}$$

ways to get the first pair and

$$\binom{12}{1}\binom{4}{2}$$

ways to get the second pair and

$$\binom{11}{1}\binom{4}{1}$$

to get the 5th card.
Putting this together we get

$$\binom{13}{1}\binom{12}{1}\binom{4}{2}^2\binom{11}{1}\binom{4}{1}$$

Can you detect the error?

1st pair is $\{2\diamondsuit, 2\clubsuit\}$, 2nd pair is $\{3\heartsuit, 3\diamondsuit\}$ and the 5th card is $\{J\clubsuit\}$.

Observe that this is the same hand as:

1st pair is $\{3\heartsuit, 3\diamondsuit\}$, 2nd pair is $\{2\diamondsuit, 2\clubsuit\}$and the 5th card is $\{J\clubsuit\}$.

So we count each hand twice the correct expression is:

$$\binom{13}{2}\binom{4}{2}^2\binom{11}{1}\binom{4}{1}$$

A poker hand is called a straight if it consists of 5 values in a row. For the purposes of this question we will exclude hands that are straight flushes.

We already know that the number of straight flushes (including royal flushes) is

$$\binom{10}{1}\binom{4}{1}$$

This assumes that all cards are of the same suit. In a straight the suit of each of the 5 cards is open to selection.

So the number of straights (including straight flushes) is:

$$\binom{10}{1}\binom{4}{1}^5$$

The final step to get the correct count is to subtract the straight flushes.

$$\binom{10}{1}\binom{4}{1}^5 - \binom{10}{1}\binom{4}{1}$$

A hand with no straight or flush or 4,3, or 2 of a kind is called a no-pair. How do we count the number of 5 card no-pair hands.

A counting idea we can exploit is:

- Count the number of ways to get 5 different cards that are not a straight.
- Count the different suits for these 5 cards that do not make a flush.

So the number of 5 card no-pair hands is:

$$\left(\binom{13}{5} - \binom{10}{1}\right)\left(\binom{4}{1}^5 - \binom{4}{1}\right)$$

The probability of getting a no-pair when randomly selecting 5 cards is:

$$\frac{\left(\binom{13}{5} - \binom{10}{1}\right)\left(\binom{4}{1}^5 - \binom{4}{1}\right)}{\binom{52}{5}}$$

And this works out to be:

(1277)(1020)/2,598,960[1] = 1,302,540/2,598,960
or about 0.5.
Expressing the odds of obtaining a no pair we get 1,296,420: 1,302,540 or 0.995:1, or almost 1:1.

You can find counting formulas for a variety of 5 card poker hands on this Wikipedia page:
https://en.wikipedia.org/wiki/Poker_probability

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[1] I used Octave, an open source version of Matlab, to compute this value.

Counting paradigms

Many of our counting problems have to do with selection. For example counting genetic sequences, that is, the number of strings of length n using the letters A C G T. One can think of this as drawing balls labelled A C G T from a bag, recording the order of the ball and replacing the ball back into the bag for subsequent selections.

This selection process can be described as

selection with ordering and replacement.

The number of ways of selecting k times from a bag of n distinct balls with ordering and replacement is:

$$k^n$$

As second example consider the number of ways to select 5 cards from a deck of 52 cards *with ordering.*

This section process can be described as

selection with ordering and without replacement

The number of ways of selecting k times from a bag of n distinct balls with ordering and without replacement is:

$$\frac{n!}{n-k!}$$

The third example to consider is the number of ways to elect 5 cards from a deck of 52 cards without ordering.

This selection process can be described as

selection without ordering and without replacement

The number of ways of selecting k times from a bag of n distinct balls without ordering and without replacement is:

$$\frac{n!}{k!(n-k)!} = \binom{n}{k}$$

Continuing this process there is one more case to consider, that is,

selection without ordering and with replacement

The next problem is an example where we use this selection process.

You get to pick a box of 10 timbits® and choose as many as you like from the choice of

Chocolate, Sugar, Plain, Glazed

The way to model this is to consider a bag with balls labelled C,S,P,G and we count the number of ways to select 10 without ordering and with replacement.

Suppose the 10 choices in order are

C,S,S,S,P,P,P,G,G,G

There are $10!/(3!)^3$ ways to order these.

On the other hand suppose the choices in order are:

C,C,C,C,C,C,C,C,C,C

There are $10!/10! = 1$ way to order this choice.

It appears that are existing methods do not solve this counting problem very easily.

Consider the following seemingly unrelated problem, that of counting the number of binary strings of length 13, consisting of 10 0’s and 3 1’s.

For example: 0100010001000

We can count the total number of this type of string as

$$13!/(3!10!)$$

Now consider a bijection from binary strings to donut selections.

I claim that there is a bijective mapping from the string

0100010001000 $\leftrightarrow$ C,S,S,S,P,P,P,G,G,G

The mapping works as follows:

The 10 0’s represent timbits®, the 1’s act as dividers partitioning the zeros into 4 groups.

What does this 0000000000111 binary string represent?
Counting

Suppose that we have n identical objects and 3 cans of paint one red, one blue, and one green. We can assume that there is enough paint in each can to colour all of the objects.

How many different ways are there to colour the objects so that each object gets only one colour?

This counting problem uses the paradigm of selecting balls from a bag without regard to ordering and replacing each ball back into the bag after it has been selected, that is,

selection without ordering and with replacement

We can model this as counting binary strings using n 0's and 2 1's. There are:

$(n+2)! / (2!\ n!)$ ways to do this. (There are n+2 symbols where 2 repeat (the 1's) and n repeat (the 0's).

Note that (n+2)! / (2! n!) can also be written as the binomial coefficient

$$\binom{n+2}{2}$$

We can think of this as a string of length n+2 of all 0’s, and we select two (different) 0’s to convert into 1’s.

Suppose that we insist that each colour is used at least once. How many ways are there to colour n identical objects with 3 colours so that each colour is used at least once.

We can think of this as pre-assigning one of the objects to each colour. So now we count the number of binary strings of length $n - 3 + 2 = n-1$ consisting of $n-3$ 0's, and 2 1's.