Read the questions carefully. Please clearly state any assumptions that you make that are not explicitly stated in the question.

Please answer all questions in the space provided. Use the back of pages for scratch work. There are 9 pages to this exam. Note that (x) denotes the question is worth x points.

1. (4) Group the following into sets that are equal.

   (a) $\emptyset$

   (b) $\{\emptyset\}$

   (c) $\{0\}$

   (d) $\{\{\}\}$

   (e) $\{\}$

2. (4) Show that there are sets A, B, C such that $A \cap B = A \cap C$ and $B \neq C$.

3. In each of the following give an example of a relation R on the set $A = \{1, 2, 3\}$ such that:

   (a) (2) R is both symmetric and antisymmetric.

   (b) (2) R is neither symmetric nor antisymmetric.

   (c) (2) R is reflexive.

(d) (2) R is a function.

(e) (2) R is an invertible function.

4. Let R be the relation on the natural numbers defined by

$$R = \{(x, y) : x, y \in \mathbb{N}, x^2 + y^2 \leq 8\}.$$

(a) (2) Write out the elements of R as a set of ordered pairs.

(b) (2) Is R an equivalence relation, and explain why or why not?

(c) (2) Is R a partial order and explain why or why not?

5. The Division Algorithm Theorem can be written as:

$$a = bq + r \text{ and } 0 \leq r < |b|$$

For each of the following statements state whether it is true or false, and if it is false correct it.

(a) (2) $a, b \in \mathbb{R}$

(b) (2) q and r are unique.

(c) (2) $q, r \in \mathbb{Z}$

6. Find the quotient q and remainder r, as given by the Division Algorithm theorem for the following examples.

(a) (2) a = 23 , b = 8

(b) (2) a = - 23, b = 8

(c) (2) a = -23, b = -8

(d) (2) a = 23, b = -8

7. Let $a = 23$, and $b = 8$. In the following $\gcd(a, b)$ and $\text{lcm}(a, b)$ respectively denote functions that return the greatest common divisor and least common multiple of a and b .

   (a) (4) Find $g = \gcd(a, b)$. Show the steps used by Euclid's algorithm to find $\gcd(a, b)$.

   (b) (4) Find integers m and n such that $g = ma + nb$, and show the steps that you used.

   (c) (4) Find $\text{lcm}(a, b)$, and show the steps that you used.

8. Let a, b, c be Integers.

   (a) (4) Prove that if $a|b$ and $a|c$, then $a|(b + c)$

   (b) (4) Prove that if $a|b$ then for any integer $n, a|bn$.

9. (5)The province of Ontario adopted the AAAA-000 licence plate format in 1997, Where an A denotes any upper case letter, [A .. Z] (there are 26 letters), and 0 denotes a digit [0 .. 9]. How many different AAAA-000 licence plates can be assigned? Explain how you arrived at your answer.

10. (5)How many different permutations of the string "AAAGTCTGAC" are there? Explain how you arrived at your answer.

11. (5) There are 30 students in a gym class. The gym teacher must partition the students into 4 teams of 9,8,7 and 6 players. In how many ways can this be done? Explain how you arrived at your answer.

12. (5) A farmer buys 3 cows, 3 pigs, and 3 hens from a farmer who has 6 cows, 5 pigs, and 8 hens. How many choices does the farmer have for selecting his new animals?

13. (5) In how many ways can 25 dimes be distributed to 4 children if each child must get at least 2 dimes? Explain how you arrived at your answer.

14. Consider the following logical expressions. Determine whether the expression is true or false, and justify your answer using truth tables.

    (a) (4) $p \to q \equiv \neg p \to \neg q$

    (b) (4) $p \to q \equiv q \to p$

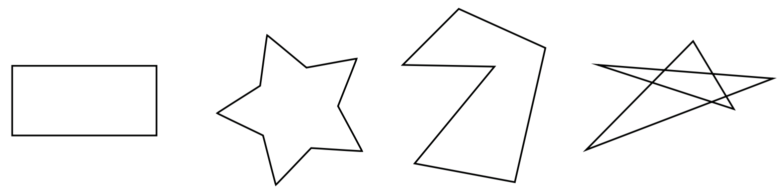
15. (5) Use induction to prove $1 + 3 + 5 + \cdots + 2n - 1 = n^2$, for all $n \in \mathbb{N}, n \geq 1$.

16. Consider the recursive function given by $a_1 = 1$ and $a_n = 3a_{n-1}$.

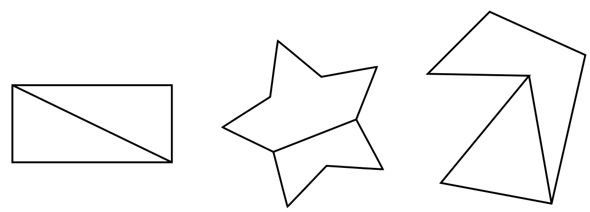
    (a) (2) Find a_2, a_3, and a_4.

    (b) (5) Using the values of a_2, a_3, and a_4 guess the value of a_n and prove that it is true using mathematical induction.

17. (5) A *simple n-gon* is a plane figure that is bounded by n line segments such that the interior of the polygon is a simply connected region. I have drawn a few examples below. The example at the far right is not a simple polygon because of self intersections and the non simply connected interior.



A *diagonal* is a line segment interior to the n-gon that partitions it into two parts, a k-gon and a j-gon, where $k + j = n + 2$. Examples are shown below.



A well known result is that one can always find a diagonal in an n-gon. Furthermore, it can be shown that non-crossing diagonals can be added to an n-gon so that the n-gon is partitioned into $n - 2$ triangles. Use induction to prove this result for all n-gons, $n \geq 4$.