CISC-102 Winter 2019

Quiz 2 // Solutions

March 7, 2019

1. (2) Let R be a relation on the set of Integers such that $(a, b) \in R$ if $b \geq a$. Which one of the following statements is false?
   (a) R is symmetric
   (b) R is antisymmetric
   (c) R is transitive
   (d) none of the above

2. (2) Let R be a relation on the set of Integers such that $(a, b) \in R$ if $b < a$. Which one of the following statements is false?
   (a) R is symmetric
   (b) R is antisymmetric
   (c) R is transitive
   (d) none of the above

3. (2) Let R be a relation on the set of Integers such that $(a, b) \in R$ if $b = a$. Which one of the following statements is false?
   (a) R is symmetric
   (b) R is antisymmetric
   (c) R is transitive
   (d) none of the above

4. (2) Which one of the following relations on the set $S = \{1, 2, 3, 4, 5, 6\}$ is a function?
   (a) R = {(1,1) (3,2), (4,2), (5,3),(6,3) }
   (b) T = {(1,1), (2,2), (3,3), (4,4) }
   (c) $S \times S$
   (d) none of the above

5. (2) Let $a = -42$, and $b = 13$ Which one of the following is the correct remainder r and quotient q when dividing a by b as given by the Division Algorithm theorem.

   (a) $r = -10, q = 4$
   (b) $r = 10, q = -4$
   (c) $r = -3, q = -3$
   (d) none of the above

6. (2) Consider two integers 102 and 18. Which of the following is true?

   (a) $\gcd(102, 18) = \gcd(18, 5)$
   (b) $\gcd(102, 18) = |\frac{102}{18}|$
   (c) $\gcd(102, 18) = \gcd(102, 5)$
   (d) none of the above

7. (2) Which of the following is false?

   (a) $c|0$ for all $c \in (Z), c \neq 0$
   (b) if $a|b$ and $b|a$ then $|a| = |b|$.
   (c) If $a|b$ then $a \leq b$
   (d) none of the above

8. (2) Which of the following is true?

   (a) The number of different strings using the letters KINGSTON is less than 8!
   (b) The number of different strings using the letters KINGSTON is greater than 8!
   (c) The number of different strings using the letters KINGSTON is equal to 8!
   (d) none of the above

9. (4) Give each of the 7 residue classes (mod 7) for integers in the range -10 to 10.

   $[0]_7$ = { -7, 0, 7}

   $[1]_7$ = { -6, 1, 8}

   $[2]_7$ = {-5,2,9}

   $[3]_7$ = {-4,3,10}

   $[4]_7$ = {-10,-3,4}

   $[5]_7$ = {-9,-2,5}

   $[6]_7$ = {-8,-1,6}

10. (4) Let $a, b, c \in \mathbb{Z}$. Prove, using the division algorithm theorem, that if $a|b$ and $a|c$, then $a|(b-c)$.

    If $a|b$ there exists an integer q such that $b = qa$. And $a|c$ implies that there is an integer p such that $c = pa$.

    Therefore, $b - c = a(q - p)$, so we conclude that $a|(b-c)$.

11. Let $a, b \in \mathbb{N}, a > b$. Prove by answering the following questions that $\gcd(a, b) = \gcd(b, a-b)$. Setting $g_1 = \gcd(a, b)$, and $g_2 = \gcd(b, a-b)$ simplifies the exposition.

    (a) (2) Prove $g_1|(a-b)$, implying that g_1 is a common divisor of b and $(a-b)$.
    Since $g_1|a$ and $g_1|b$, we have $g_1|(a-b)$.

    (b) (2) Prove $g_2|a$, implying that g_2 is a common divisor of a and b.
    Since $g_2|(a-b)$ and $g_2|b$, we have $g_2|(a-b)+b$, therefore $g_2|a$.

    (c) (2) Prove $g_1 \geq g_2$ and $g_2 \geq g_1$, implying $\gcd(a, b) = \gcd(b, a-b)$.
    Part a) implies that $g_2 \geq g_1$, and part b) implies that $g_1 \geq g_2$. Therefore, we conclude that $g_1 = g_2$.

12. (4) Let S be a finite subset of the positive integers. What is the smallest value for $|S|$ that guarantees that at least two elements $x, y \in S$ have the same remainder when divided by 42. HINT: Use the pigeon hole principle.

    There are 42 distinct remainders when dividing by 42, or said another way, exactly 42 distinct residue classes. Therefore, by the pigeon hole principle any subset of the natural numbers with at least 43 elements must have two or more elements within the same residue class.

13. (6) Consider the recursive function $F(1) = 1, F(2) = 2, F(n) = F(n-1) + F(n-2)$, for integers $n \geq 3$.

    Use the second form of Mathematical induction to prove that $F(n) \leq 2^n$, for all $n \in \mathbb{N}$.

    Base: $F(1) = 1 \leq 2^1$, and $F(2) = 2 \leq 2^2$

    Induction Hypothesis: Assume that $F(j) \leq 2^j$ for $j, 1 \leq j \leq k$.

    Induction Step: $F(k+1) = F(k) + F(k-1) \leq 2^k + 2^{k-1} \leq 2(2^k) = 2^{k+1}$.

14. Let $g = \gcd(a, b)$.

    (a) (2) Prove that $\frac{a}{g}$ and $\frac{b}{g}$ are integers.
    g divides both a and b so $\frac{a}{g}$ and $\frac{b}{g}$ are integers.

(b) (3) Prove that $\gcd(\frac{a}{g}, \frac{b}{g}) = 1$.

Let $f = \gcd(\frac{a}{g}, \frac{b}{g})$. Therefore, there exists integers x, y such that
$\frac{a}{g} = fx$ which implies that $a = fgx$
and $\frac{b}{g} = fy$ which implies that $b = fgy$.
Therefore, fg is a common divisor of a and b, and since $g = \gcd(a, b)$, we conclude that $f = 1$.