

CISC-102 Winter 2019

Homework 2 Solutions

Problems

1. Illustrate DeMorgan's Law $(A \cap B)^c = A^c \cup B^c$ using Venn diagrams.

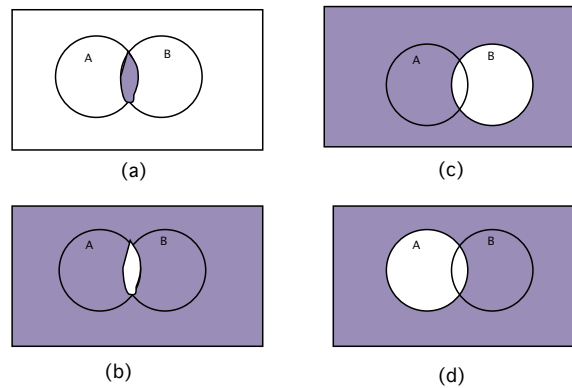


Figure 1: $(A \cap B)$ is shown in (a), and (c) and (d) illustrate B^c and A^c respectively. Finally (b) shows that $(A \cap B)^c = A^c \cup B^c$

2. Let $A_i = \{1, 2, 3, \dots, i\}$ for all $i \in \mathbb{N}$. For example $A_4 = \{1, 2, 3, 4\}$.

- (a) What are the elements of the set $\cup_{i=1}^n A_i$?

$$\bigcup_{i=1}^n A_i = A_n$$

- (b) What are the elements of the set $\cap_{i=1}^n A_i$?

$$\bigcap_{i=1}^n A_i = A_1$$

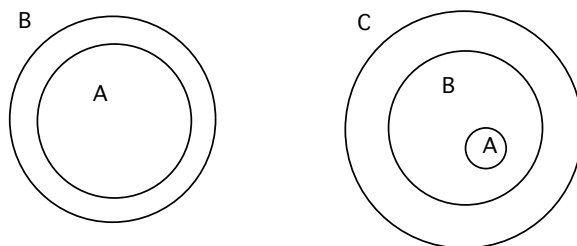


Figure 2: $A \subseteq B$ is shown on the left, and $A \subseteq B \subseteq C$ is shown on the right.

3. Observe that $A \subseteq B$ has the same meaning as $A \cap B = A$. Draw a Venn diagram to illustrate this fact.

See Figure 2. If $A \subseteq B$ then every element $x \in A$ is also an element in B , which in turn implies that $A \cap B = A$.

4. Use a Venn diagram to show that if $A \subseteq B$ **and** $B \subseteq C$, then $A \subseteq C$.

See Figure 2. $A \subseteq B$ implies that every element of A is also in B , $x \in A$ implies $x \in B$. Similarly $B \subseteq C$ implies that every element of B is also in C , $y \in B$ implies $y \in C$. Thus $A \subseteq C$.

5. Use the Principle of Exclusion and Inclusion to show that $|A \cup B| + |A \cap B| = |A| + |B|$. (It may help your understanding if you first explore an example such as $A = \{1,2,3\}$ and $B = \{3,4\}$).

By the Principle of Inclusion Exclusion we have $|A| + |B| - |A \cap B| = |A \cup B|$. These quantities are just non-negative integers so if we add $|A \cap B|$ to the right and left side of the equation, we get the desired result.

6. What are the cardinalities of the following sets?

- (a) $A = \{\text{winter, spring, summer, fall}\}$. $|A| = 4$.
- (b) $B = \{x : x \in \mathbb{Z}, 0 < x < 7\}$. $|B| = 6$.
- (c) $P(B)$, that is, the power set of B . $|P(B)| = 2^6 = 64$.
- (d) $C = \{x : x \in \mathbb{N}, x \text{ is even}\}$ This set has infinitely many elements.

7. Suppose that we have a sample of 100 students at Queen's who take at least one of the following language courses, French-101, Spanish-101, German-101. Also suppose that 65 take French-101, 45 take German-101, 42 take Spanish 101, 20 take French-101 and German-101, 25 take French-101 and Spanish-101, and 15 take German-101 and Spanish-101.

- (a) How many students take all three language courses?

Let F , S , and G denote the sets of students taking French Spanish and German respectively. The Principle of Inclusion and Exclusion tells us that

$$|F \cup S \cup G| = |F| + |S| + |G| - |F \cap S| - |S \cap G| - |F \cap G| + |F \cap S \cap G|$$

The problem statement gives us values for each quantity in the equation except for $|F \cap S \cap G|$. We can now simply fill in the numbers and solve for $|F \cap S \cap G|$, as follows:

$$100 = 65 + 42 + 45 - 25 - 15 - 20 + |F \cap S \cap G|$$

So we conclude that $|F \cap S \cap G| = 8$

- (b) Draw a Venn diagram representing these 100 students and fill in the regions with the correct number.
- (c) How many students take exactly 1 of these courses? Using the Venn diagram we can deduce that $28+10+18 = 56$ students take exactly one of the language courses.
- (d) How many students take exactly 2 of these courses? Using the Venn diagram we can deduce that $17 + 12 + 7 = 36$ students take exactly two courses.

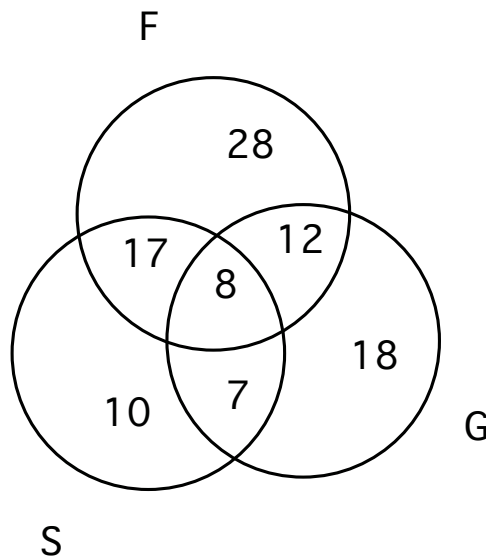


Figure 3: Language Courses Venn Diagram

- 8. At an art class with 30 students, there are 14 women, and 16 men. Twenty-two of the students are right-handed. What is the minimum and maximum number of women that are right-handed?

With all of the men are right handed there must be 8 right handed women as the minimum. The maximum is when all of the women are right handed, that is 14.

9. Recall that the union operation is associative, that is $A \cup (B \cup C) = (A \cup B) \cup C$. Show that the relative complement set operation is not associative, that is, $A \setminus (B \setminus C) = (A \setminus B) \setminus C$, is incorrect for some sets A, B, C . (Note if relative complement is associative then the equation must be true for all sets A, B, C .)

Let $A = \{1,2,3\}$ $B = \{1,2\}$ and $C = \{2,3\}$. $A \setminus (B \setminus C) = \{2,3\}$ and $(A \setminus B) \setminus C = \emptyset$

10. Consider a set S of n elements, such that $\{a, b\} \subseteq S$.

- (a) What is the cardinality of the power set of $S \setminus \{a\}$?

We know that S has n elements so $S \setminus \{a\}$ has $n - 1$ elements. The power set of $S \setminus \{a\}$ has 2^{n-1} elements.

- (b) What is the cardinality of the power set of $S \setminus \{a, b\}$?

The power set of $S \setminus \{a, b\}$ has 2^{n-2} elements.

- (c) How many subsets of S are there that contain the element a ?

Here is a way to construct the subsets of S that contain the element a . For each subset $s \in P(S \setminus \{a\})$ construct the set $\{a\} \cup s$. This yields all subsets of S that contain a . Since there are 2^{n-1} subsets of $S \setminus \{a\}$, there are 2^{n-1} subsets of S that contain the element a .

We can obtain the same result by using a different argument. We know that there are 2^n subsets of S . The subsets of $S \setminus \{a\}$ are also subsets of S that do not contain a . The total number of subsets of S is 2^n . So the number of subsets of S that contain a is equal to $2^n - 2^{n-1} = 2^{n-1}$.

- (d) How many subsets of S are there that contain the element a and exclude the element b ?

Here is a way to construct the subsets of S that contain a and exclude b . For each subset $s \in P(S \setminus \{a, b\})$ construct the set $\{a\} \cup s$. This yields 2^{n-2} subsets of S .