CISC-102 WINTER 2019

HOMEWORK 3
SOLUTIONS

(1) Prove using mathematical induction that the sum of the first n natural numbers is equal to $\frac{n(n+1)}{2}$. This can also be stated as:

Prove that the proposition P(n),

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

is true for all $n \in \mathbb{N}$.

Base: for n =1, $1 = \frac{1(1+1)}{2}$

Induction hypothesis: Assume that P(k) is true, that is:

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}.$$

for $k \geq 1$.

Induction step: The goal is to show that P($k+1$) is true, that is:

$$\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}.$$

Consider the sum

$$\begin{aligned}\sum_{i=1}^{k+1} i = &\sum_{i=1}^{k} i + (k+1)(\text{arithmetic})\\ =&\frac{k(k+1)}{2} + (k+1)(\text{Use the induction hypothesis})\\ =&\frac{k^2+k+2k+2}{2}(\text{get common denominator and add})\\ =&\frac{k^2+3k+2}{2}(\text{add k + 2k})\\ =&\frac{(k+1)(k+2)}{2}(\text{factor to arrive at goal})\end{aligned}$$

We have shown that P(k) true implies that P($k+1$) is true so by the principle of mathematical induction we conclude that P(n) is true for all $n \in \mathbb{N}$. $\square$

(2) Prove using mathematical induction that the proposition P(n),

$$\sum_{i=2}^{n} i = \frac{(n-1)(n+2)}{2}$$

is true for all $n \in \mathbb{N}, n \geq 2$

Base: for $n = 2$, $2 = \frac{1(2+2)}{2}$

Induction hypothesis: Assume that P(k) is true, that is:

$$\sum_{i=2}^{k} i = \frac{(k-1)(k+2)}{2}.$$

for $k \geq 2$.

Induction step: The goal is to show that P($k+1$) is true, that is:

$$\sum_{i=2}^{k+1} i = \frac{(k)(k+3)}{2}.$$

Consider the sum

$$\begin{aligned}\sum_{i=2}^{k+1} i &= \sum_{i=2}^{k} i + (k+1)\text{(arithmetic)}\\ &= \frac{(k-1)(k+2)}{2} + (k+1)\text{(Use the induction hypothesis)}\\ &= \frac{k^2+k-2+2k+2}{2}\text{(get common denominator and add)}\\ &= \frac{k^2+3k}{2}\text{(arithmetic)}\\ &= \frac{k(k+3)}{2}\text{(factor to arrive at goal)}\end{aligned}$$

We have shown that P(k) true implies that P($k+1$) is true so by the principle of mathematical induction we conclude that P(n) is true for all $n \in \mathbb{N}, n \geq 2$. □

(3) Prove using mathematical induction that the proposition P(n),

$$\sum_{i=3}^{n} i = \frac{(n-2)(n+3)}{2}$$

is true for all $n \in \mathbb{N}, n \geq 3$

Base: for $n = 3$, $3 = \frac{(3-2)(3+3)}{2}$

Induction hypothesis: Assume that P(k) is true, that is:

$$\sum_{i=3}^{k} i = \frac{(k-2)(k+3)}{2}.$$

for $k \geq 3$.

Induction step: The goal is to show that P($k+1$) is true, that is:

$$\sum_{i=3}^{k+1} i = \frac{(k-1)(k+4)}{2}.$$

Consider the sum

$$\begin{aligned}
\sum_{i=3}^{k+1} i = \sum_{i=3}^{k} i + (k+1)&\text{(arithmetic)}\\
&= \frac{(k-2)(k+3)}{2} + (k+1)\text{(Use the induction hypothesis)}\\
&= \frac{k^2+k-6+2k+2}{2}\text{(get common denominator and add)}\\
&= \frac{k^2+3k-4}{2}\text{(arithmetic)}\\
&= \frac{(k-1)(k+4)}{2}\text{(factor to arrive at goal)}
\end{aligned}$$

We have shown that P(k) true implies that P($k+1$) is true so by the principle of mathematical induction we conclude that P(n) is true for all $n \in \mathbb{N}, n \geq 3$. □

(4) Prove using mathematical induction that the proposition P(n)

$$n! \leq n^n$$

is true for all $n \in \mathbb{N}$.

Base: for n =1, $1! = 1 = 1^1$

Induction hypothesis: Assume that P(k) is true, that is:

$$k! \leq k^k$$

for $k \geq 1$.

Induction step: The goal is to show that P($k+1$) is true, that is:

$$(k+1)! \leq (k+1)^{k+1}.$$

We have:

$$\begin{aligned}(k+1)! =& k!(k+1) \text{(Definition of factorial)}\\ \leq& k^k(k+1) \text{(Use the induction hypothesis)}\\ \leq& (k+1)^k(k+1) \text{(because } k \leq k+1)\\ =& (k+1)^{k+1} \text{(multiply)}\end{aligned}$$

We have shown that P(k) true implies that P($k+1$) is true so by the principle of mathematical induction we conclude that P(n) is true for all $n \in \mathbb{N}$. □

(5) Let P(n) be the proposition that if $A_1, A_2, \ldots A_n, B_1, B_2, \ldots, B_n$ are sets such that $A_i \subseteq B_i$ for all $i, 1 \leq i \leq n$, then $\cap_{i=1}^n A_i \subseteq \cap_{i=1}^n B_i$. Prove, using mathematical induction that P(n) is true for all natural numbers $n \geq 2$.

Base The set $A_1 \cap A_2 = \{x : x \in A_1 \text{ and } x \in A_2\}$. If $A_i \subseteq B_i$ for $i = 1, 2$ then every element of $A_1 \cap A_2$ is also an element of B_1 and of B_2. Since $B_1 \cap B_2 = \{x : x \in B_1 \text{ and } x \in B_2\}$, we conclude that $A_1 \cap A_2 \subseteq B_1 \cap B_2$.

Induction Hypothesis Assume that that if $A_1, A_2, \ldots A_k, B_1, B_2, \ldots, B_k$ are sets such that $A_i \subseteq B_i$ for all $i, 1 \leq i \leq k$, then $\cap_{i=1}^k A_i \subseteq \cap_{i=1}^k B_i$.

Induction Step. Consider $2(k+1)$ sets, $A_1, A_2, \ldots A_{k+1}, B_1, B_2, \ldots, B_{k+1}$, such that $A_i \subseteq B_i$ for all $i, 1 \leq i \leq k+1$. We can partition these sets into $A_1, A_2, \ldots A_k, B_1, B_2, \ldots, B_k$ and A_{k+1}, B_{k+1}. Applying the induction hypothesis we have $\cap_{i=1}^k A_i \subseteq \cap_{i=1}^k B_i$. For the induction step we only need to argue about pairs of sets, just as we did in the base case. So we have the set $\cap_{i=1}^k A_i \cap A_{k+1} = \{x : x \in \cap_{i=1}^k A_i \text{ and } x \in A_{k+1}\}$. If $\cap_{i=1}^k A_i \subseteq \cap_{i=1}^k B_i$ and $A_{k+1} \subseteq B_{k+1}$ then every element of $\cap_{i=1}^k A_i \cap A_{k+1}$ is also an element of $\cap_{i=1}^k B_i$ and of B_{k+1}. Since $\cap_{i=1}^k B_i \cap B_{k+1} = \{x : x \in \cap_{i=1}^k B_i \text{ and } x \in B_{k+1}\}$, we conclude that $\cap_{i=1}^{k+1} A_i \subseteq \cap_{i=1}^{k+1} B_i$.

We have shown that P(k) true implies that P($k+1$) is true so by the principle of mathematical induction we conclude that P(n) is true for all $n \in \mathbb{N}, n \geq 2$. □

(6) Given a set of n points on a two dimensional plane, such that no three points are on the same line, it is always possible to connect every pair of points with a line segment. The figure illustrates this showing 5 points, that are pairwise connected with 10 line segments. Prove using mathematical induction that the total number of line segments is $\frac{n(n-1)}{2}$ for any number of points $n \in \mathbb{N}, n \geq 2$.

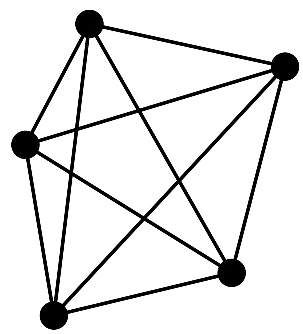

FIGURE 1. Five points, pairwise connected with 10 line segments.

Base: Given two points there is exactly one segment that connects them.

Induction Hypothesis: Assume that k points can be connected by $\frac{k(k-1)}{2}$ line segments for some fixed natural numer $k, k \geq 2$.

Induction Step: Consider $k+1$ points. We can partition the points into two subsets with k points in one and a single point in the other. The induction hypothesis implies that there are $\frac{k(k-1)}{2}$ line segments connecting the k points. The $k+1^{st}$ point can now be connected to these k points with k line segments. Therefore we have $\frac{k(k-1)}{2}+k = \frac{(k+1)k}{2}$ line segments connecting all $k+1$ points.

We have shown that $\mathrm{P}(k)$ true implies that $\mathrm{P}(k+1)$ is true so by the principle of mathematical induction we conclude that $\mathrm{P}(n)$ is true for all $n \in \mathbb{N}, n \geq 2$. $\square$