CISC-102 WINTER 2019

HOMEWORK 8
SOLUTIONS

(1) How many ways are there to select a 5 card poker hand from a standard deck of 52 cards, such that none of the cards are clubs? We make the selection using 52-13 =

39 club free cards. Thus we have

$$\binom{39}{5}$$

ways to make the selection.

(2) How many ways are there to select a 5 card poker hand poker hand from a standard deck of 52 cards, such that at least one of the cards is a club?

We subtract selections with no clubs from all possible selections, yielding the following expression:

$$\binom{52}{5} - \binom{39}{5}$$

(3) A skip straight is 5 cards that are in consecutive order, skipping every second rank (for example 3-5-7-9-J). How many 5 card hands are there (unordered selection from a standard 52 card deck) that form a skip straight?

One card defines a skip straight. That is if you know that the smallest value in the skip straight is a 3, then the other four values are fixed. A skip straight's lowest value comes from the set {A, 2, 3, 4, 5, 6 }. (We assume that Ace can be either low = 1, or high = 14. Once the numbers are fixed each of the cards can be any of the four suits. So the total number of skip straights is $4^5 \times 6$. This can also be written as:

$$\binom{4}{1}^5\binom{6}{1}$$

(4) You are planning a dinner party and want to choose 5 people to attend from a list of 11 close personal friends.

(a) In how many ways can you select the 5 people to invite.

$$\binom{11}{5}$$

(b) Suppose two of your friends are a couple and will not attend unless the other is invited. How many different ways can you invite 5 people under these constraints?

$$\binom{9}{3} + \binom{9}{5}$$

The first binomial coefficient in the sum counts selections with the couple, and the second counts selections without the couple.

(c) Suppose two of your friends are enemies, and will not attend unless the other is not invited. How many different ways can you invite 5 people under these constraints?

$$\binom{9}{5} + 2\binom{9}{4}$$

The first binomial coefficient in the sum counts selections without the enemies. The second counts selections with either enemy A, or enemy B.

(5) What is the number of ways to colour n different objects, one colour per object with 2 colours? What is the number of ways to colour n different objects with 2 colours, so that each colour is used at least once.

There are 2^n ways of colouring the n different objects using 2 colours. There exactly two ways where only one colour is used. So there are $2^n - 2$ ways to colour n different objects with 2 colours, so that each colour is used at least once.

(6) What is the number of ways to colour n identical objects with 3 colours? What is the number of ways to colour n identical objects with 3 colours so that each colour is used at least once?

The number of ways to colour n objects with 3 colours can be viewed as counting the number of binary strings with n 0's and 2 1's. This yields the expression:

$$\frac{(n+2)!}{n!2!} = \binom{n+2}{2} = \binom{n+2}{n}$$

To ensure that each colour is used at least once we pre-assign one object per colour leaving $n-3$ objects to be coloured with no further restrictions. We map this problem to counting binary strings with $n-3$ 0's and 2 1's. This yields the expression:

$$\frac{(n-3+2)!}{(n-3)!2!} = \binom{n-1}{2} = \binom{n-1}{n-3}$$

(7) Consider the equation

$$x_1 + x_2 + x_3 + x_4 = 7$$

A non-negative integer solution to this equation assigns non-negative integers (integers $x, x \geq 0$) to the variables x_1, x_2, x_3, x_4 so that the sum is 7. For example one possible solution is $x_1 = 1, x_2 = 3, x_3 = 1, x_4 = 2$. And another distinct solution is $x_1 = 2, x_2 = 3, x_3 = 1, x_4 = 1$ How many distinct non-negative integer solutions are there to this equation?

We count binary strings with 3 1's and 7 0's. The first example solution is encoded as 0100010100 and the second solution as 0010001010.

The solution is:

$$\frac{10!}{3!7!} = \binom{10}{3} = \binom{10}{7}$$

(8) From 100 used cars siting on a lot, 20 are to be selected for a test designed to check safety requirements. These 20 cars will be returned to the lot, and again 20 will be selected for testing for emission standards.

(a) In how many ways can the cars be selected for safety requirement testing?
$\binom{100}{20}$

(b) In how many ways can the cars be selected for emission standards testing?
$\binom{100}{20}$

(c) In how many different ways can the cars be selected for both tests?
$\binom{100}{20}\binom{100}{20}$

(d) In how many ways can the cars be selected for both tests if exactly 5 cars must be tested for safety and emission?
$\binom{100}{5}\binom{95}{15}\binom{80}{15}$

(9) Use the binomial theorem to expand the product $(x + y)^6$.

Recall: The binomial theorem can be stated as:

$$(x + y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i.$$

So for this question we have:

$$(x+y)^6 = \binom{6}{0}x^6 + \binom{6}{1}x^5y + \binom{6}{2}x^4y^2 + \binom{6}{3}x^3y^3 + \binom{6}{4}x^2y^4 + \binom{6}{5}xy^5 + \binom{6}{6}y^6.$$

(10) Show that

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n\binom{n}{n} = 0$$

HINT: Use the Binomial theorem.
Note that this equation can also be written as follows:

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^i = 0$$

The binomial theorem with $a = 1$ and $b = -1$ can be written as:

$$0 = (1-1)^n = \sum_{i=0}^{n}\binom{n}{i}(1^{n-i})(-1)^i$$

And this proves the result.