CISC-102 WINTER 2019

HOMEWORK 9
SOLUTIONS

(1) Consider the equation

$$\sum_{i=0}^{2} \binom{3}{i}\binom{2}{2-i} = \binom{5}{2}. \tag{1}$$

(a) Use algebraic manipulation to prove that the left hand and right hand sides of equation (1) are in fact equal.

First we work out the left hand side of equation (1).

$$\begin{aligned}
\sum_{i=0}^{2} \binom{3}{i}\binom{2}{2-i} &= \binom{3}{0}\binom{2}{2} + \binom{3}{1}\binom{2}{1} + \binom{3}{2}\binom{2}{0} \\
&= 1 + 6 + 3 \\
&= 10
\end{aligned}$$

And the right hand side of equation (1).

$$\begin{aligned}
\binom{5}{2} &= \frac{5!}{3!2!} \\
&= \frac{5 \times 4}{2} \\
&= 10
\end{aligned}$$

(b) Use a counting argument to prove that the left hand and right hand sides of equation (1) are in fact equal.

On the right we count the number of ways of selecting 2 balls from a bag of 5 different balls without regard to ordering.

On the left we have two bags one with 3 balls and the other with 2 balls, which we call the 3bag and 2bag respectively. We now sum the products of selecting, without ordering, 0 from the 3bag times 2 from the 2bag, 1 from the 3bag and 1 from the 2bag, and 2 from the 3 bag and 0 from the 2 bag.

(2) Now consider a generalization of the previous equation.

$$\sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i} = \binom{m+n}{k}. \tag{2}$$

Use a counting argument to prove that the left hand and right hand sides of equation (2) are in fact equal.

On the right we count the number of ways of selecting k balls without ordering from a bag of $m+n$ balls. On the left we count selections from two bags one with m balls and the other with n balls. We sum products of selecting $k-i$ and i balls from the two bags.

(3) In the notes for Week 9 you will find Pascal's triangle worked out for rows 0 to 8. The numbers in row 8 are 1 8 28 56 70 56 28 8 1. Work out the values of rows 9 and 10 of Pascal's triangle with the help of the equation:

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}. \tag{3}$$

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row 8:     1   8   28   56   70   56   28    8    1
row 9:  1   9   36    84   126 126  84    36   9   1
row10:1   10 45   120   210 252  210   120 45 10   1
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(4) Show that $\binom{n}{0} = \binom{n-1}{0}$, and that $\binom{n-1}{n-1} = \binom{n}{n}$ by an algebraic argument as well as a counting argument.

Recall that $0! = 1$. So we have:

$$\binom{n}{0} = \frac{n!}{n!0!} = 1,$$

and

$$\binom{n-1}{0} = \frac{(n-1)!}{(n-1)!0!} = 1.$$

The counting argument is that choosing nothing from any number of items is always 1, and in particular for n items and $n-1$ items.

$$\binom{n}{n} = \frac{n!}{n!0!} = 1,$$

and

$$\binom{n-1}{n-1} = \frac{(n-1)!}{(n-1)!0!} = 1.$$

The counting argument is that there is only one way to choose all items, and in particular for n items and $n-1$ items.

(5) Prove that

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n \binom{n}{n} = 0$$

Note that this equation can also be written as follows:

$$\sum_{i=0}^{n} \binom{n}{i} (-1^i) = 0$$

HINT: This can be viewed as a special case of the binomial theorem.

Observe that by the binomial theorem we have:

$$0 = (1-1)^n = \sum_{i=0}^{n} \binom{n}{i} (-1^i) 1^{n-i}$$

And this proves the result.

(6) Prove that:

$$\frac{n^k}{k^k} \leq \binom{n}{k} \leq \frac{n^k}{k!}$$

Note: All you need is a bit of algebra for this one.

Observe that:

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{n}{k} \times \frac{n-1}{k-1} \times \cdots \times \frac{n-(k-1)}{k-(k-1)} = \prod_{m=0}^{k-1} \frac{n-m}{k-m}$$

and

$$\frac{n^k}{k^k} = \frac{n}{k} \times \frac{n}{k} \times \cdots \times \frac{n}{k} = \prod_{m=0}^{k-1} \frac{n}{k}$$

We now compare $\frac{n}{k}$ to $\frac{n-m}{k-m}$ where $0 \leq m \leq k-1$, and $k < n$.
Observe that under these conditions we have

$$n(k-m) \leq k(n-m)$$

Now divide the left and right hand side of this inequality by $k(k-m)$ to get:

$$\frac{n}{k} \leq \frac{n-m}{k-m}$$

So we may conclude that:

$$\prod_{m=0}^{k-1} \frac{n}{k} \leq \prod_{m=0}^{k-1} \frac{n-m}{k-m}.$$

We now show that

$$\binom{n}{k} \leq \frac{n^k}{k!}$$

We resort to the product notation again to immediately see that

$$\prod_{m=0}^{k-1} \frac{n-m}{k-m} \leq \prod_{m=0}^{k-1} \frac{n}{k-m}. \quad \square$$

(7) Let $F(n)$ denote the n^{th} Fibonacci number. Prove, using mathematical induction, that

$$\sum_{i=1}^{n} F(2i-1) = F(2n)$$

For all natural numbers n.

Base: (For $n = 1$)$F(2-1) = F(2)$

Induction Hypothesis: Assume that $\sum_{i=1}^{k} F(2i-1) = F(2k)$

Induction Step:

$$\begin{aligned}\sum_{i=1}^{k+1} F(2i-1) &= \sum_{i=1}^{k} F(2i-1) + F(2(k+1)-1) \\ &= F(2k) + F(2k+1) \\ &= F(2(k+1)) \quad \square\end{aligned}$$

(8) Prove, using mathematical induction, that

$$\sum_{i=1}^{n} F(i)^2 = F(n)F(n+1)$$

For all natural numbers n.

Base:(For $n = 1$)$F(1)^2 = F(1)F(2)$

Induction Hypothesis: Assume that $\sum_{i=1}^{k} F(i)^2 = F(k)F(k+1)$

Induction Step:

$$\begin{aligned}\sum_{i=1}^{k+1} F(i)^2 &= \sum_{i=1}^{k} F(i)^2 + F(k+1)^2 \\ &= F(k)F(k+1) + F(k+1)^2 \\ &= F(K+1)(F(k) + F(k+1)) \\ &= F(k+1)F(k+2). \quad \square\end{aligned}$$