CISC-102
Winter 2019
Week 5

Relations (See chapter 2. of SN)

Functions are mappings from one set to another with specific additional properties.

Recall: A function must map every element of the Domain set to a single element in the Range set.

Mappings without these additional properties are also valid entities in mathematics.

An ordered pair of elements a,b is written as (a,b).
NOTE: Mathematical convention distinguishes between "( )" brackets -order is important – and "{ }" -- not ordered.
Example: $\{1,2\} = \{2,1\}$, but $(1,2) \neq (2,1)$.

Product Sets

Let A and B be two arbitrary sets. The set of all ordered pairs (a,b) where $a \in A$ and $b \in B$ is called the product or Cartesian product or cross product of A and B.
The cross product is denoted as:

$$A \times B = \{(a,b) : a \in A \text{ and } b \in B\}$$

and is pronounced “A cross B”.
It is common to denote $A \times A$ as A^2.

A “famous” example of a product set is , $\mathbb{R}^2$, that is, the product of the Reals, or the two dimensional real plane or Cartesian plane -- x and y coordinates.

Relations

Definition: Let A and B be arbitrary sets. A *binary relation*, or simply a *relation* from A to B is a subset of
$A \times B$.
(We study relations to continue our exploration of mathematical definitions and notation.)

Example: Suppose $A = \{1,3,6\}$ and $B = \{1,4,6\}$

$$A \times B = \{(a,b) : a \in A, \text{ and } b \in B \}$$
$$= \{(1,1),(1,4),(1,6),(3,1),(3,4),(3,6),(6,1),(6,4)(6,6)\}$$

Example: Consider the relation $\leq$ on $A \times B$ where A and B are defined above.
The subset of $A \times B$ in this relation are the pairs:
$\{(1,1),(1,4),(1,6),(3,4),(3,6),(6,6)\}$

That is, a pair (a,b) is in the relation $\leq$ whenever $a \leq b$.

Vocabulary

When we have a relation on $S \times S$ (which is a very common occurrence) we simply call it a relation *on* S, rather than a relation on $S \times S$.

Let $A = \{1,2,3,4\}$, we can define the following relations on A.

$R_1 = \{(1,1), (1,2), (2,3), (1,3), (4,4)\}$

$R_2 = \{(1,1), (1,2), (2,1), (2,2), (3,3), (4,4)\}$

$R_3 = \{(1,3), (2,1)\}$

$R_4 = \varnothing$

$R_5 = A \times A = A^2$ (How many elements are there in R_5 ?)

Properties of relations on a set A

Reflexive: A relation R is <u>reflexive</u>
if $(a,a) \in R$ for all $a \in A$.

Symmetric: A relation R is <u>*symmetric*</u>
if whenever $(a_1, a_2) \in R$ then $(a_2, a_1) \in R$.

Antisymmetric: A relation R is <u>*antisymmetric*</u>
if whenever $(a_1, a_2) \in R$ and $(a_2, a_1) \in R$ then $a_1 = a_2$.

NOTE: There are relations that are neither symmetric nor antisymmetric or both symmetric and antisymmetric.

Transitive: A relation R is <u>*transitive*</u>
if whenever $(a_1, a_2) \in R$ and $(a_2, a_3) \in R$ then $(a_1, a_3) \in R$.

Let $A = \{1,2,3,4\}$, we can define the following relations on A.
$R_1 = \{(1,1), (1,2), (2,3), (1,3), (4,4)\}$
NOT reflexive, NOT symmetric, antisymmetric, transitive

$R_2 = \{(1,1), (1,2), (2,1), (2,2), (3,3), (4,4)\}$
reflexive, symmetric, NOT antisymmetric, transitive

$R_3 = \{(1,3), (2,1)\}$
NOT reflexive, NOT symmetric, antisymmetric, NOT transitive
$R_4 = \varnothing$
NOT reflexive, symmetric, antisymmetric, transitive

$R_5 = A \times A = A^2$ (How many elements are there in R_5 ?)
reflexive, symmetric, transitive.
Consider the relation

$R_6 = \{(1,1), (1,2), (2,1), (2,3),(2,2), (3,3)\}$
NOT reflexive, NOT symmetric, NOT antisymmetric, NOT transitive

Consider the relations $<$, $\leq$, and $=$ on the Natural numbers. (less than, less than or equal to, equal to)
The relation $<$ on the Natural numbers

$\{(a,b) : a,b \in N, a < b\}$ is:

NOT reflexive, NOT symmetric, antisymmetric, transitive

The relation $\leq$ is on the Natural numbers

$\{(a,b) : a,b \in N, a \leq b\}$ is:

reflexive, NOT symmetric, antisymmetric, transitive

The relation $=$ on the Natural numbers

$\{(a,b) : a,b \in N, a = b\}$ is:

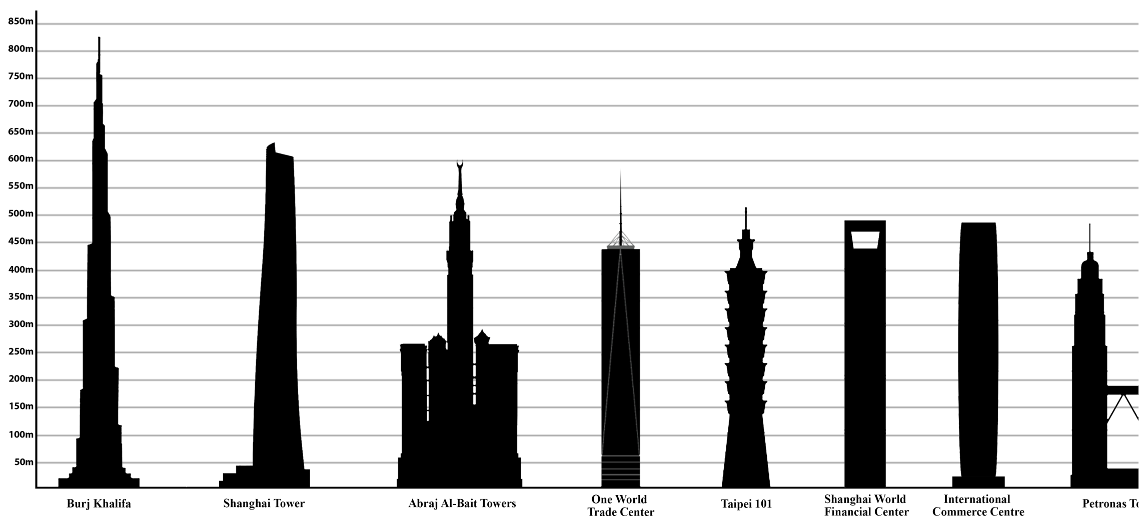
reflexive, symmetric, antisymmetric, transitive

Partial orders and equivalence relations

A relation R is called a *partial order* if R is reflexive, antisymmetric, and transitive.

Partial order relations can be used when we want to compare and order things.

NOTE: The relation $\leq$ is on the Natural numbers
$\{(a,b) : a,b \in N, a \leq b\}$ is a partial order relation.



We can order the tallest buildings in the world by height.

Let P(S) denote the power set of the set S, and let R be a relation on P(S) defined as:

$R = \{(s,t) \in P(S) \times P(S) : s \subseteq t \}$

Observe that R is a partial order, because:
$(s,s) \in R$ for all sets $s \in P(S)$, therefore R is reflexive.

Whenever $s \subseteq t$ and $t \subseteq s$, then $s = t$, therefore R is antisymmetric

Whenever $s \subseteq t$ and $t \subseteq w$, then $s \subseteq w$, therefore R is transitive.

A relation R is called an *equivalence relation* if R is reflexive, symmetric, and transitive.

Equivalence relations can be used when we want to compare and classify things.
The relation = on the Natural numbers
$\{(a,b) : a,b \in N, a = b\}$ is an equivalence relation.

We can partition fruit into *equivalence classes* using an equivalence relation.

Suppose R is an equivalence relation on a set S. For each element $s \in S$, let $[s] = \{t \in S : (s,t) \in R\}$. We call [s] an *equivalence class* of S.

For example let S be the set {A,B,C, a,b,c,1,2,3} and let R be the relation $\{(s,t) \in S \times S$: s and t are both upper case, both lower case, or both digits}.

Thus, R partitions S into 3 equivalence classes,
$[a] = \{a,b,c\}$, $[A] = \{A,B,C\}$, $[1] = \{1,2,3\}$.

Observe that:
$(s,s) \in R$ so R is reflexive.
Whenever $(s,t) \in R$, then $(t,s) \in R$.
Whenever $(s,t) \in R$ and $(t,v) \in R$, then $(s,v) \in R$.

So R is an equivalence relation. Furthermore, note that

$[a] \cap [A] = \emptyset$, $[a] \cap [1] = \emptyset$, $[A] \cap [1] = \emptyset$, and that
$[a] \cup [A] \cup [1] = S$.

That is the equivalence classes partition the set S.

Functions as relations

A function can be viewed as a special case of relations.

A relation R from A to B is a function if every element $a \in A$ belongs to a unique ordered pair (a,b) in R.

Let A = {a,b,c, …, z} and let S = {1,2,3, …, 26}. We define a relation R from A to S as:
$R = \{(x,y) \in A \times S$: letter x is the y^{th} letter of the alphabet.}

We can verify that R is a function be observing that for every letter $x \in A$, there is a single value $y \in S$, such that $(x,y) \in R$

In fact R is a bijection, that is, a one-to-one and onto function. Why?

Also observe that $|A| = |S|$.