CISC-102
Winter 2019
Week 7

Congruence Relations

Let a and b be integers. We say that a is congruent to b modulo m written as:

$a \equiv b \pmod{m}$

and defined as follows:

$a \equiv b \pmod{m}$ if and only if $m \mid (a-b)$.

Consider two integers a and b whose difference is a multiple of m.

1-2-3-4-5-6-7-8-9-10-11-12-13-14-15-16-17-18-19-20-21-22-23

Observe that a and b have the same remainder when divided by m.

Suppose m | (a-b) therefore a-b = pm for some integer p.

And:

(1) a = b + pm

We can also express b as an integer (call it q) multiple of m plus a remainder:

(2) b = qm + r

Putting (1) and (2) together we get

a = b + pm = qm + r + pm = m(p+q) + r

So we conclude that if m | (a-b) then a and b have the same remainder when divided by m.

Now suppose a and b are integers that have the same remainder when divided by m.

We have:

a = xm + r and b = ym + r,

where x and y are integers and r is the common remainder.

Therefore a - b = m(x-y), so m | (a-b).

Congruences we use everyday:

1:00 PM = 13:00 o'clock, that is, $1 \equiv 13 \pmod{12}$

The numbers 24 and 42 are even, that is $24 \equiv 42 \pmod{2}$

Notes on a piano keyboard. $\pmod{12}$

The name of the month 14 months ago is the same as the name of the month 2 months ago. (mod 12)

The name of the day 10 days ago is the same as the name of the day 3 days ago. (mod 7)

Congruence modulo m is an equivalence relation. Observe that we can partition the integers by their congruences.

Examples:

Congruence (mod 2) partitions integers into those that are even and odd.

Congruence (mod 3) partitions integers into three classes those that are divisible by 3 (remainder 0) and those with remainder 1, and remainder 2 when divided by 3.

In general we say that congruence modulo m partitions the integers into m classes called *residue classes modulo m*. Furthermore, each of these residue classes can be denoted by an integer x within the class using the notation $[x]_m$. Using set notation we can express this as follows:

$$[x]_m = \{a \in \mathbb{Z} : a \equiv x \bmod m\}$$

And each of the residue classes can be denoted by its smallest member as follows:

$$[0]_m, [1]_m, [2]_m, \ldots, [m-1]_m$$

Recall an equivalence relation is reflexive, symmetric, and transitive.

We can verify that congruence is an equivalence relation.

Reflexive $a \equiv a \pmod{m}$ for all integers a,
because $m \mid (a-a)$.

Symmetric if $a \equiv b \pmod{m}$ then $b \equiv a \pmod{m}$,
because if $m \mid (a-b)$ then $m \mid -1(a-b)$
or $m \mid (b-a)$.

Transitive if $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$
then $a \equiv c \pmod{m}$.
because if $m \mid (a-b)$ and $m \mid (b-c)$
then $m \mid ((a-b) + (b-c))$ or $m \mid (a - c)$

Arithmetic with congruences

Suppose we have $a \equiv b \pmod m$ and $c \equiv d \pmod m$.

Then

$a + c \equiv (b + d) \pmod m$,

$a - c \equiv (b - d) \pmod m$, and

$ac \equiv (bd) \pmod m$.

Examples

$5 \equiv 2 \pmod 3$ and $10 \equiv 1 \pmod 3$

$5 + 10 \equiv (2 + 1) \pmod 3$, that is, $15 \equiv 3 \pmod 3$

$5 - 10 \equiv (2 - 1) \pmod 3$, that is, $-5 \equiv 1 \pmod 3$

(Note: By the Division Algorithm Theorem we have $-5 = (-2)(3) + 1$)

$(5)(10) \equiv (2)(1) \pmod 3$, that is, $50 \equiv 2 \pmod 3$

Suppose we have $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$.
Then $a + c \equiv (b + d) \pmod{m}$.

Proof: (We need to show that $a + c \equiv (b + d) \pmod{m}$.)

If $a \equiv b \pmod{m}$ then $m \mid (a-b)$.
And if $c \equiv d \pmod{m}$ we have $m \mid (c-d)$.

This in turn implies that

$$m \mid ((a - b) + (c - d))$$

which can be written as

$$m \mid ((a + c) - (b + d)).$$

So we can conclude that $a + c \equiv (b + d) \pmod{m}$. □

Suppose we have $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$.
Then $ac \equiv (bd) \pmod{m}$. □

Proof: (We need to show that $m \mid (ac - bd)$.)

If $a \equiv b \pmod{m}$ then $m \mid (a-b)$.
And if $c \equiv d \pmod{m}$ we have $m \mid (c-d)$.

This in turn implies that
$m \mid (a - b)c$ (because $m \mid (a - b)p$ for all integers p)
and that
$m \mid (c - d)b$ (because $m \mid (a - b)p$ for all integers p).

Therefore we have

$$m \mid ((a - b)c + (c - d)b)$$

Which can be written as:

$$m \mid (ac - bd)$$

So we can conclude that $ac \equiv (bd) \pmod{m}$. □

Techniques of Counting (Chapter 5 of SN)

We have already seen and solved several counting problems.
For example:

- How many subsets are there of a set with n elements?
- How many two element subsets are there of a set with n elements.

Counting problems are useful to determine resources used by an algorithm (*e.g.* time and space).

Product Rule Principle

Let $A \times B$ denote the cross product of sets A and B.
Then $|A \times B| = |A| \times |B|$[1]

For example suppose you have to pick a main course from: Fish, Beef, Chicken, Vegan. We can write this as the set M (Main), as follows

$$M = \{F,B,C,V\}$$

Furthermore there is also choice of a desert from: Apple pie, Lemon meringue pie, Ice cream. This can be represented as the set D.

$$D = \{A,L,I\}$$

When we select a meal we select a main course AND a desert.
We use the product rule to determine the total number of possible meals, that is:

$$|\{F,B,C,V\}| \times |\{A,L,I\}| = (4)(3) = 12.$$

[1] Recall: vertical bars represent cardinality, or the number of elements in the set.

The **product rule principle** can be stated formally as:

Suppose there is an event E that occurs in m ways and an event F that occurs in n ways, and these two events are *independent* of each other. Then the combination the events E AND F can occur in $m \times n$ ways.

Product Rule Principle

The rule generalizes to any number of independent sets (events). For example with 3 sets:
Let A × B × C denote the cross product of sets A, B, & C.

Then |A × B × C | = |A| × |B| ×|C| .

For k sets we have:

$$|A_1 \times A_2 \times \ldots \times A_k| = |A_1| \cdot |A_2| \cdot \ldots \cdot |A_k|$$

For example, DNA is represented using the 4 symbols:

A C G T.

The number of different strings of length 7 using these symbols is:

$$4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 = 4^7.$$

The number of strings of length k using these 4 symbols is:

$$4^k$$

Sum Rule Principle

Suppose we have the same mains and deserts as before, and we can also choose a soup or a salad.

Where the soups are:

$$S = \{\text{Ministrone, Lobster Bisque, Tomato}\}$$

And the salads are

$$T = \{ \text{ Garden, Caesar } \}$$

In how many ways can we choose a soup OR a salad?

We have $|S| = 3$ and $|T| = 2$ for a total of $3 + 2 = 5$ choices.

Note that these sets have an empty intersection. For non-empty intersections we would need to use the principle of inclusion and exclusion.

The **sum rule** can be stated formally as:

Suppose some event E can occur *m* ways, and a second event F can occur in *n* ways, and the two events do not occur at once, then E OR F can occur in *m* + *n* ways.

And sometime we combine the two principles. As in counting the number of meals we can make when choosing

3 Soups OR 2 Salads

AND

4 Mains

AND

3 Deserts

is: (3 + 2) (4) (3) = 60 different meals.

The Pigeon Hole Principle

If there are $n+1$ pigeons, that all must sleep in a pigeon hole, and n pigeon holes, then there is at least one pigeon hole where (at least) 2 pigeons sleep.

This should be obvious! Mathematicians give it a name because it is a useful counting tool.

Do two people exist who live in the G.T.A. and have exactly the same number of strands of hair on their heads?

The answer is YES! And we can prove it using the pigeon hole principle.

The population of the G.T.A is more than 6 million. Science tells us that nobody has more that 500,000 strands of hair on their heads.

To solve the problem using the pigeon hole principle we imagine 500,000 pigeon holes labelled from 1, ..., 500,000 and then imagine each resident of the G.T.A. entering the pigeon hole labelled with the number of strands of hair on their head. Since 6 million is greater than 500,001 we deduce that there will be at least one pigeon hole where two or more people have entered.

The generalized pigeonhole principle

Let k be a positive integer.

If there are $kn+1$ pigeons, that all must sleep in a pigeon hole, and n pigeon holes, then there is at least one pigeon hole where (at least) $k+1$ pigeons sleep.

Observe that 6,000,000 = 12 * 500,00, so we can conclude that there exists at least 12 + 1 = 13 people that live in the G.T.A. with the same number of strands of hair on their heads.

Can we find 2 pairs of people living in the G.T.A. that have exactly the same number of strands of hair on their heads?

The pigeon hole principle is useless for solving this problem and we leave this as an unsolved mystery.

Let's look at two more applications of the pigeon hole principle.

Find the minimum number n of integers to be selected from S = {1, 2, …, 9} so that the sum of two of the integers is guaranteed to be even.

If a number x is odd then $x = 2p + 1$ for some integer p. And similarly an odd number y yields, $y = 2q + 1$ for some integer q. Thus $x + y = 2(p+q + 1)$ and is divisible by two. Similarly one can show that the sum of 2 even numbers is even.

This leads to the observation that as long as we have two odd or two even integers we get an even sum, so we partition S into even and odd numbers. By the pigeon hole principle 3 numbers from S will always contain a pair that sums to an even number.

Pigeon holes are: {1,3,5,7,9} and {2,4,6,8}

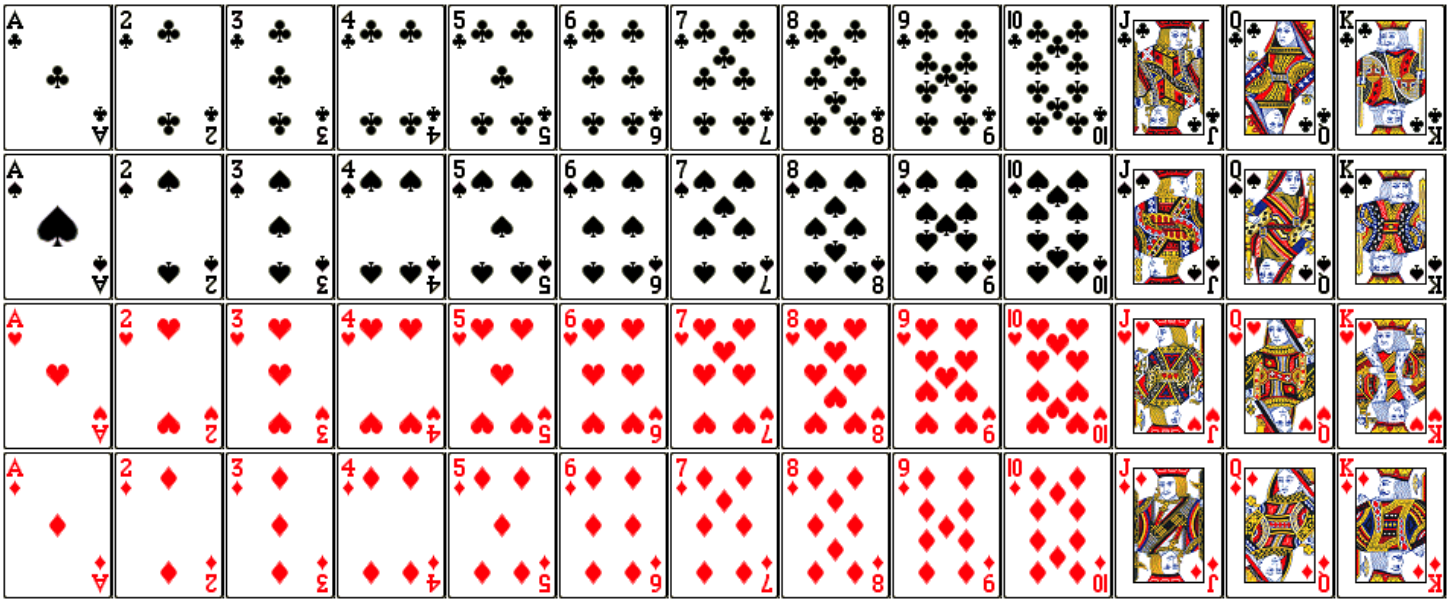
Find the minimum number n of integers to be selected from S = {1, 2, …, 9} so that the absolute difference between two of the integers is exactly 5.

We partition S into pairs that yield a difference of 5.

Pigeon holes are: {1,6},{2,7},{3,8},{4,9},{5}

So we need to pick 6 numbers to guarantee that difference of two is 5.

Playing cards.



Some of the following examples make use of the standard 52 deck of playing cards as shown below.

There are 4 suits (clubs, spades, hearts, diamonds) each consisting of 13 values (Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, King) for a total of 52 cards.

Permutations

A common paradigm for counting is to imagine selecting labeled balls from a bag, so that no two balls are alike.

A permutation of objects is represented by a record of the order in which balls are pulled out of the bag.

Example: How many ways are there to select 5 different coloured balls from a bag?

$$5 \times 4 \times 3 \times 2 \times 1 = 5!$$

We can relate this to the product rule by thinking of the full bag as the set B_5, the bag with 4 balls as the set B_4, the bag with 3 balls B_3, the bag with 2 balls B_2, and with 1 ball B_1. Thus pulling balls from a bag can be viewed as a combination of the events (sets of outcomes) B_1, B_2, B_3, B_4, B_5. And the number of ways the combination of these events can occur as:

$$|B_1| \times |B_2| \times |B_3| \times |B_4| \times |B_5| = 5 \times 4 \times 3 \times 2 \times 1 = 5!$$

Example: How many different ways are there to shuffle a deck of cards?

We can number the cards in a deck from 1 to 52 where 1 is the card on top and 52 is the card on the bottom. So shuffling a deck of cards is equivalent to assigning a unique number from 1 ... 52 to each of the cards.

Observe that there is a bijection between the number of ways to draw balls from a bag, and the number of ways to select positions in a shuffled deck of cards. There are 52 positions to select as represented by the the following expression.

$$52 \times 51 \times 50 \ldots \times 1 = 52!$$

A permutation of the elements of a set is in essence assigning an ordering to a set.

Permutation rule

There are n! ways to permute n elements.

Example

Larry has 6 distinguishable pairs of socks. Each day Monday to Saturday he wears a different pair of socks. On Sunday he washes the socks (and goes sock-less). In how many different ways can Larry wear a week's worth of socks?

Permutation of a Subset

Suppose we want to count the number of ways of selecting 2 coloured balls from a total of 5 coloured balls.

$$5 \times 4 = 5!/3!$$

Suppose we want to count the number of ways to make an ordered selection of just 5 of the 52 cards.

$$52 \times 51 \times 50 \times 49 \times 48 = 52!/47!$$

different ways.

NOTATION:

$$P(n,k) = n!/(n-k)!$$

represents the number of permutations of k elements chosen from a collection of n elements.

Using our Poker hand analogy, a 5 card poker hand drawn from a 52 card deck one at a time, where order is taken into account has:

$52 \times 51 \times 50 \times 49 \times 48 = 52!/(52-5)! = 52!/47!$

different ways of occurring.

Permutations with Repetition

How many different ways can we order the letters**:**

BABY?

You may be temped to say 4! = 24 different ways, (that is select 4 balls labelled B A B Y from a bag) but upon inspection we see that there are only 12 distinguishable ways to order the letters.

The list of all 24 permutations that you see come in pairs.

BABY BABY	BYAB BYAB	AYBB AYBB
BAYB BAYB	BYBA BYBA	YBBA YBBA
BBAY BBAY	ABYB ABYB	YBAB YBAB
BBYA BBYA	ABBY ABBY	YABB YABB

I used colour to distinguish between the two B's in BABY. However, in reality the two B's are not distinguishable, and the list really should look like:

BABY BABY	BYAB BYAB	AYBB AYBB
BAYB BAYB	BYBA BYBA	YBBA YBBA
BBAY BBAY	ABYB ABYB	YBAB YBAB
BBYA BBYA	ABBY ABBY	YABB YABB

The correct way to count this is 4!/2! because two of the letters in B A B Y are identical.

How many ways are there to order the letters CCCB?

BCCC
CBCC
CCBC
CCCB

There are 4!/3! = 4 ways

How many ways are there to order the letters BBCC?

BBCC
BCBC
BCCB
CBBC
CBCB
CCBB

There are 4!/2!2! = 6 ways

Example: How many ways are there to pick ten coloured balls from a bag where each colour appears twice, so that two balls of the same colour are indistinguishable?

$$\frac{10!}{2!2!2!2!2!} = \frac{10!}{(2!)^5}$$

The counting formula is: The number of permutations of n objects consisting of $n_1, n_2, n_3, \ldots, n_r$ that are alike is:

$$\frac{n!}{n_1!n_2!\ldots n_r!}$$

Combinations

Suppose on the other hand that we want to count the number of different 5 card poker hands. We are interested in the number of ways of selecting 5 from 52 without regard to the way that they are ordered. We can solve this counting problem by answering the following questions.

(1) How many ways are there to shuffle a 5 card deck?

Answer: 5!

(2) How many ways are there to make an ordered selection of 5 of the 52 cards?

Answer: 52!/47!

(3) How do we put these two answers together to count the number of ways to make an un-ordered selection of 5 of the 52 cards?

Answer: We divide the answer to (2) by the answer to (1), yielding: 52!/(47!5!).

Combinations

We can use the balls in a bag analogy to count combinations. In this case we count the number of different ways to select distinct balls without ordering. The counting technique is a 2 step process.

1. Count the number of ways to select k balls from a bag of n balls with ordering.
2. Divide by the number of ways to order the k selected balls.

The outcome of this process yields the formula:

$$\frac{n!}{(n-k)!k!}$$

We have seen this expression before and the accompanying shorthand, that is:

$$\frac{n!}{(n-k)!k!} = \binom{n}{k}$$

NOTATION: $C(n,k) = P(n,k)/k! = \binom{n}{k}$