Counting Poker Hands.

Notation:

A card from a standard 52 card deck will be denoted using an ordered pair as follows:

(v,s) where v is an element of the set of 13 values:

$$\{A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, K, Q\}.$$

and s is an element of the set of 4 suits:

$$\{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}.$$

For this discussion a poker hand is a 5 card subset of the 52 card deck.

There are 2,598,960 different 5 card subsets from a 52 card deck.

How is this value obtained?

The most valuable poker hand is a royal flush, that is a 5 card subset that consists of the values 10,J,Q,K,A all in the same suit.

For example:

{(10,♧), (J,♧), (Q,♧), (K, ♧), (A,♧)}

is one example of a royal flush.

How many royal flushes are there?

There are exactly 4 different royal flushes. The “odds” of obtaining a royal flush is expressed as a ratio of all non royal flush 5 card poker hands versus royal flushes.

This ratio is:

(2,598,960-4):4

And simplifies to 649,739 : 1.

The next highest hand is a straight flush. That is a hand of 5 consecutive values (where A = 1 or A =14 as appropriate) all of the same suit. Normally the designation straight flush excludes the royal flushes.

There are a total of

$$\binom{10}{1}\binom{4}{1} - \binom{4}{1}$$

straight flushes.

A four of a kind consists of 4 cards of the same value plus one additional card.

Let's look at two equivalent ways of counting the number of 4 of kind hands.

1. Count the number of ways to select the value of the four of a kind (13) and then the number of ways to choose the 5th card (48), and multiply.
2. Count the number of ways to select the value of the four of a kind (13) and then number of ways to select the suit of the value of the 5th card (12) and the number of ways to select the suit of the 5th card (4), and multiply.

The odds of getting a four of a kind is 4,164 : 1

To see why we compute the product:

13(48) = 624

So there are 2,598,960 - 624 = 2,598,336 ways to get a "non four of a kind" vs. 624 ways to get a 4 of a kind, giving:

2,598,336:624 odds

which simplifies to:

4,164 : 1

A full house consists of 3 cards of the same value plus 2 cards of the same value?

For example: $\{7\clubsuit, 7\diamondsuit, 7\spadesuit, 3\heartsuit, 3\spadesuit\}$ is a full house.

There are:

$$\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}$$

ways to get a full house.

A poker hand is called 3 of a kind, when 3 cards have the same value, and the other two can be any two of the remaining values.

For example: $\{7\clubsuit, 7\diamondsuit, 7\spadesuit, 2\heartsuit, 3\spadesuit\}$ makes 3 of a kind.

How many different 3 of a kind hands are there?

A poker hand is called two pair if it consists of two distinct pairs of the same value and a 5th card with value different from the first two.

An <u>incorrect</u> way to count this is:

There are

$$\binom{13}{1}\binom{4}{2}$$

ways to get the first pair and

$$\binom{12}{1}\binom{4}{2}$$

ways to get the second pair and

$$\binom{11}{1}\binom{4}{1}$$

to get the 5th card.
Putting this together we get

$$\binom{13}{1}\binom{12}{1}\binom{4}{2}^2\binom{11}{1}\binom{4}{1}$$

Can you detect the error?

1st pair is $\{2\diamondsuit, 2\clubsuit\}$, 2nd pair is $\{3\heartsuit, 3\diamondsuit\}$ and the 5th card is $\{J\clubsuit\}$.

Observe that this is the same hand as:

1st pair is $\{3\heartsuit, 3\diamondsuit\}$, 2nd pair is $\{2\diamondsuit, 2\clubsuit\}$and the 5th card is $\{J\clubsuit\}$.

So we count each hand twice the correct expression is:

$$\binom{13}{2}\binom{4}{2}^2\binom{11}{1}\binom{4}{1}$$

A poker hand is called a straight if it consists of 5 values in a row. For the purposes of this question we will exclude hands that are straight flushes.

We already know that the number of straight flushes (including royal flushes) is

$$\binom{10}{1}\binom{4}{1}$$

This assumes that all cards are of the same suit. In a straight the suit of each of the 5 cards is open to selection.

So the number of straights (including straight flushes) is:

$$\binom{10}{1}\binom{4}{1}^5$$

The final step to get the correct count is to subtract the straight flushes.

$$\binom{10}{1}\binom{4}{1}^5 - \binom{10}{1}\binom{4}{1}$$

A hand with no straight or flush or 4,3, or 2 of a kind is called a no-pair. How do we count the number of 5 card no-pair hands.

A counting idea we can exploit is:

- Count the number of ways to get 5 different cards that are not a straight.
- Count the different suits for these 5 cards that do not make a flush.

So the number of 5 card no-pair hands is:

$$\left(\binom{13}{5} - \binom{10}{1}\right)\left(\binom{4}{1}^5 - \binom{4}{1}\right)$$

The probability of getting a no-pair when randomly selecting 5 cards is:

$$\frac{\left(\binom{13}{5} - \binom{10}{1}\right)\left(\binom{4}{1}^5 - \binom{4}{1}\right)}{\binom{52}{5}}$$

And this works out to be:

(1277)(1020)/2,598,960[1] = 1,302,540/2,598,960
or about 0.5.
Expressing the odds of obtaining a no pair we get 1,296,420: 1,302,540 or 0.995:1, or almost 1:1.

You can find counting formulas for a variety of 5 card poker hands on this Wikipedia page:
https://en.wikipedia.org/wiki/Poker_probability

[1] I used Octave, an open source version of Matlab, to compute this value.

Counting paradigms

Many of our counting problems have to do with selection. For example counting genetic sequences, that is, the number of strings of length n using the letters A C G T. One can think of this as drawing balls labelled A C G T from a bag, recording the order of the ball and replacing the ball back into the bag for subsequent selections.

This selection process can be described as

selection with ordering and replacement.

The number of ways of selecting k times from a bag of n distinct balls with ordering and replacement is:

$$k^n$$

As second example consider the number of ways to select 5 cards from a deck of 52 cards *with ordering.*

This section process can be described as

selection with ordering and without replacement

The number of ways of selecting k times from a bag of n distinct balls with ordering and without replacement is:

$$\frac{n!}{n-k!}$$

The third example to consider is the number of ways to elect 5 cards from a deck of 52 cards without ordering.

This selection process can be described as

selection without ordering and without replacement

The number of ways of selecting k times from a bag of n distinct balls without ordering and without replacement is:

$$\frac{n!}{k!(n-k)!} = \binom{n}{k}$$

Continuing this process there is one more case to consider, that is,

selection without ordering and with replacement

The next problem is an example where we use this selection process.

You get to pick a box of 10 timbits® and choose as many as you like from the choice of

Chocolate, Sugar, Plain, Glazed

The way to model this is to consider a bag with balls labelled C,S,P,G and we count the number of ways to select 10 without ordering and with replacement.

Suppose the 10 choices in order are

C,S,S,S,P,P,P,G,G,G

There are $10!/(3!)^3$ ways to order these.

On the other hand suppose the choices in order are:

C,C,C,C,C,C,C,C,C,C

There are $10!/10! = 1$ way to order this choice.

It appears that are existing methods do not solve this counting problem very easily.

Consider the following seemingly unrelated problem, that of counting the number of binary strings of length 13, consisting of 10 0's and 3 1's.

For example: 0100010001000

We can count the total number of this type of string as

$$13!/(3!10!)$$

Now consider a bijection from binary strings to donut selections.

I claim that there is a bijective mapping from the string

0100010001000 ↔ C,S,S,S,P,P,P,G,G,G

The mapping works as follows:

The 10 0's represent timbits®, the 1's act as dividers partitioning the zeros into 4 groups.

What does this 0000000000111 binary string represent?
Counting

Suppose that we have n identical objects and 3 cans of paint one red, one blue, and one green. We can assume that there is enough paint in each can to colour all of the objects.

How many different ways are there to colour the objects so that each object gets only one colour?

This counting problem uses the paradigm of selecting balls from a bag without regard to ordering and replacing each ball back into the bag after it has been selected, that is,

selection without ordering and with replacement

We can model this as counting binary strings using n 0's and 2 1's. There are:

$(n+2)! / (2!\ n!)$ ways to do this. (There are n+2 symbols where 2 repeat (the 1's) and n repeat (the 0's).

Note that (n+2)! / (2! n!) can also be written as the binomial coefficient

$$\binom{n+2}{2}$$

We can think of this as a string of length n+2 of all 0's, and we select two (different) 0's to convert into 1's.

Suppose that we insist that each colour is used at least once. How many ways are there to colour n identical objects with 3 colours so that each colour is used at least once.

We can think of this as pre-assigning one of the objects to each colour. So now we count the number of binary strings of length $n - 3 + 2 = n-1$ consisting of $n-3$ 0's, and 2 1's.

The Binomial Theorem

When we expand the expression:
$(x + y)^3$

we get:
$(x+y)(x+y)(x+y) = x^3 + 3x^2y + 3xy^2 + y^3$

this can also be written as follows:

$$(x+y)(x+y)(x+y) = \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3$$

We can reason that when we expand *$(x + y)^3$,* there is one way to choose a triple that is exclusively x's (with 0 y's), 3 ways to choose a triple that has 2 x's (and 1 y) , and 3 ways to choose a triple that has 1 x (and 2 y's). Finally there is 1 way to choose a triple with no x (and 3 y's).

Binomial Theorem:

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n}x^0 y^n$$
$$= \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k$$

For all natural numbers *n*.

Proof: In the expansion of the product:

$$(x + y)\,(x + y) \cdots (x+y),$$

there $\binom{n}{k}$ ways to choose an *n*-tuple with n-*k* *x*'s and (*k* *y*'s). $\square$

A special case of the binomial theorem should look familiar.

$$
\begin{aligned}
(1+1)^n &= \binom{n}{0}1^n1^0 + \binom{n}{1}1^{n-1}1 + \binom{n}{2}1^{n-2}1^2 + \cdots + \binom{n}{n}1^0 1^n \\
&= \sum_{k=0}^{n} \binom{n}{k}
\end{aligned}
$$

This is just the sum the sizes of
all subsets of a set of size *n*.

Using counting to prove theorems.

Counting arguments can be useful tool for proving theorems. In each case there is also an algebraic way of proving the result. However, there is an inherent beauty in the elegant simplicity of some of these counting arguments so it's well worth looking at some examples. These proofs lack the formality of algebraic proofs. The lack of formality may make these arguments harder to grasp for some, and easier to understand for others.

The proofs we see will be to prove the validity of equations. We will count the left and right hand side of each equation and show that they count the same thing.

Binomial Coefficients

We prove identities involving binomial coefficients using counting arguments.

Theorem:

$$\binom{n}{k} = \binom{n}{n-k}$$

Proof: On the left we have the quantity $\binom{n}{k}$ which represents the number of ways to select a *k* element subset from an *n* element set, *S*. Using the analogy of selecting balls from a bag, we see that we also implicitly select the complementary subset that stays in the bag, and the number of ways to do this is as given on the right hand side of the equation is $\binom{n}{n-k}$. $\square$

Theorem A:

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

Proof: On the left the quantity $\binom{n+1}{k}$ represents the number of ways to select a k element subset from an $n+1$ element set. To see what the right hand side counts we suppose that there is a "favourite" or "distinguished" element of the set, call it x.

The number of ways to select a k element subset from $n+1$ distinct objects that is guaranteed to <u>*include*</u> x is to pull x out and then choose the remaining $k-1$ elements in $\binom{n}{k-1}$ ways. On the other hand the number of ways to select a k element subset from $n+1$ distinct objects that is guaranteed to <u>*exclude*</u> x is to pull x out and then choose all k elements in $\binom{n}{k}$ ways.

Therefore the left and right hand side both count the same thing thus justifying the equation. □

And here's an alternate algebraic proof.

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$$

Proof:

$$\begin{aligned}
\binom{n}{k-1} + \binom{n}{k} &= \frac{n!}{(n-k+1)!(k-1)!} + \frac{n!}{(n-k)!(k)!} \\
&= \frac{n!k + n!(n-k+1)}{(n+1-k)!k!} \\
&= \frac{n!(k+n-k+1)}{(n+1-k)!k!} \\
&= \frac{n!(n+1)}{(n+1-k)!k!} \\
&= \frac{(n+1)!}{(n+1-k)!k!} \\
&= \binom{n+1}{k}
\end{aligned}$$

Theorem:

$$\sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$$

Proof: On the left the sum counts all the subsets of a set of size n. We already know that the number of subsets of a set of size n, is 2^n.

Therefore the left and right hand side both count the same thing thus justifying the equation. □