CISC-102
Winter 2019
Week 9

Binomial Theorem:

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n}x^0 y^n$$
$$= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

For all natural numbers *n*.

Proof: In the expansion of the product:

(x + y) (x + y) ··· (x+y),

there $\binom{n}{k}$ ways to choose an *n*-tuple with n-*k* *x*'s and (*k* *y*'s). $\square$

A special case of the binomial theorem should look familiar.

$$\begin{aligned}(1+1)^n &= \binom{n}{0}1^n1^0 + \binom{n}{1}1^{n-1}1 + \binom{n}{2}1^{n-2}1^2 + \cdots + \binom{n}{n}1^0 1^n \\ &= \sum_{k=0}^{n}\binom{n}{k}\end{aligned}$$

This is just the sum the sizes of
all subsets of a set of size *n*.

Using counting to prove theorems.

Counting arguments can be useful tool for proving theorems. In each case there is also an algebraic way of proving the result. However, there is an inherent beauty in the elegant simplicity of some of these counting arguments so it's well worth looking at some examples. These proofs lack the formality of algebraic proofs. The lack of formality may make these arguments harder to grasp for some, and easier to understand for others.

The proofs we see will be to prove the validity of equations. We will count the left and right hand side of each equation and show that they count the same thing.

Binomial Coefficients

We prove identities involving binomial coefficients using counting arguments.

Theorem:

$$\binom{n}{k} = \binom{n}{n-k}$$

Proof: On the left we have the quantity $\binom{n}{k}$ which represents the number of ways to select a *k* element subset from an *n* element set, *S*. Using the analogy of selecting balls from a bag, we see that we also implicitly select the complementary subset that stays in the bag, and the number of ways to do this is as given on the right hand side of the equation is $\binom{n}{n-k}$. $\square$

Theorem A:

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

Proof: On the left the quantity $\binom{n+1}{k}$ represents the number of ways to select a k element subset from an $n+1$ element set. To see what the right hand side counts we suppose that there is a "favourite" or "distinguished" element of the set, call it x.
The number of ways to select a k element subset from $n+1$ distinct objects that is guaranteed to *include* x is to pull x out and then choose the remaining $k-1$ elements in $\binom{n}{k-1}$ ways. On the other hand the number of ways to select a k element subset from $n+1$ distinct objects that is guaranteed to *exclude* x is to pull x out and then choose all k elements in $\binom{n}{k}$ ways.

Therefore the left and right hand side both count the same thing thus justifying the equation. □

And here's an alternate algebraic proof.

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$$

Proof:

$$\begin{aligned}
\binom{n}{k-1} + \binom{n}{k} &= \frac{n!}{(n-k+1)!(k-1)!} + \frac{n!}{(n-k)!(k)!} \\
&= \frac{n!k + n!(n-k+1)}{(n+1-k)!k!} \\
&= \frac{n!(k+n-k+1)}{(n+1-k)!k!} \\
&= \frac{n!(n+1)}{(n+1-k)!k!} \\
&= \frac{(n+1)!}{(n+1-k)!k!} \\
&= \binom{n+1}{k}
\end{aligned}$$

Theorem:

$$\sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$$

Proof: On the left the sum counts all the subsets of a set of size n. We already know that the number of subsets of a set of size n, is 2^n.

Therefore the left and right hand side both count the same thing thus justifying the equation. □

Pascal's Triangle

An easy way to calculate a table of binomial coefficients was recognized centuries ago by mathematicians in India, China, Iran and Europe.
In the west the technique is named after the French mathematician Blaise Pascal (1623-1662). In the example below each row represents the binomial coefficients as used in the binomial theorem.

$$
\begin{array}{ccccccccccccc}
 & & & & & & \binom{0}{0} & & & & & & \\
 & & & & & \binom{1}{0} & & \binom{1}{1} & & & & & \\
 & & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & & \\
 & & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & & \\
 & & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & & \\
 & \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} & \\
\binom{6}{0} & & \binom{6}{1} & & \binom{6}{2} & & \binom{6}{3} & & \binom{6}{4} & & \binom{6}{5} & & \binom{6}{6}
\end{array}
$$

To obtain the entries by hand in a simple way we can use the identity:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

$$\begin{array}{ccccccccccccc}
 & & & & & & \binom{0}{0} & & & & & & \\
 & & & & & \binom{1}{0} & & \binom{1}{1} & & & & & \\
 & & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & & \\
 & & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & & \\
 & & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & & \\
 & \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} & \\
\binom{6}{0} & & \binom{6}{1} & & \binom{6}{2} & & \binom{6}{3} & & \binom{6}{4} & & \binom{6}{5} & & \binom{6}{6}
\end{array}$$

$$\begin{array}{ccccccccccccccccc}
 & & & & & & & & 1 & & & & & & & & \\
 & & & & & & & 1 & & 1 & & & & & & & \\
 & & & & & & 1 & & 2 & & 1 & & & & & & \\
 & & & & & 1 & & 3 & & 3 & & 1 & & & & & \\
 & & & & 1 & & 4 & & 6 & & 4 & & 1 & & & & \\
 & & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & & \\
 & & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & & \\
 & 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 & \\
1 & & 8 & & 28 & & 56 & & 70 & & 56 & & 28 & & 8 & & 1
\end{array}$$

Consider the sum of elements in a row of Pascal's triangle. If we label the top row 0, then it appears that row i sums to the value 2^i. Can you explain why this is the case?

$$
\begin{array}{ccccccccccccc}
 & & & & & & \binom{0}{0} & & & & & & \\
 & & & & & \binom{1}{0} & & \binom{1}{1} & & & & & \\
 & & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & & \\
 & & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & & \\
 & & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & & \\
 & \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} & \\
\binom{6}{0} & & \binom{6}{1} & & \binom{6}{2} & & \binom{6}{3} & & \binom{6}{4} & & \binom{6}{5} & & \binom{6}{6}
\end{array}
$$

$$
\begin{array}{ccccccccccccccccc}
 & & & & & & & & 1 & & & & & & & & \\
 & & & & & & & 1 & & 1 & & & & & & & \\
 & & & & & & 1 & & 2 & & 1 & & & & & & \\
 & & & & & 1 & & 3 & & 3 & & 1 & & & & & \\
 & & & & 1 & & 4 & & 6 & & 4 & & 1 & & & & \\
 & & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & & \\
 & & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & & \\
 & 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 & \\
1 & & 8 & & 28 & & 56 & & 70 & & 56 & & 28 & & 8 & & 1
\end{array}
$$

Now let's compute the sum of squares of the entries of each row in Pascal's triangle.

$1^2 = 1$
$1^2 + 1^2 = 2$
$1^2 + 2^2 + 1^2 = 6$
$1^2 + 3^2 + 3^2 + 1^2 = 20$
$1^2 + 4^2 + 6^2 + 4^2 + 1^2 = 70$

These sums all appear in the middle row of Pascal's triangle.

$$
\begin{array}{ccccccccccccccccc}
&&&&&&&&1&&&&&&&& \\
&&&&&&&1&&1&&&&&&& \\
&&&&&&1&&2&&1&&&&&& \\
&&&&&1&&3&&3&&1&&&&& \\
&&&&1&&4&&6&&4&&1&&&& \\
&&&1&&5&&10&&10&&5&&1&&& \\
&&1&&6&&15&&20&&15&&6&&1&& \\
&1&&7&&21&&35&&35&&21&&7&&1& \\
1&&8&&28&&56&&70&&56&&28&&8&&1
\end{array}
$$

Which leads us to conjecture that:

$$\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}$$

Before proving the theorem there are two preliminary lemmas.

Lemma 1:

$$\binom{n}{k}\binom{n}{n-k} = \binom{n}{k}^2$$

For all non-negative integers *n,k, n* > *k*.

Proof: Since we already showed that $\binom{n}{k} = \binom{n}{n-k}$ this should be obvious. □

$$\sum_{i=0}^{k} \binom{m}{k-i} \binom{n}{i} = \binom{m+n}{k}$$

Lemma 2:

For all non-negative integers *m,n,k* such that $n \geq m \geq k$.

Proof: We use a counting argument. The right hand side can be viewed as the number of subsets of size *k* chosen from the union of two disjoint sets, *S* of size *m*, and *T* of size *n*. On the left we sum the choices where all *k* are from *S*, then *k-1* from *S* and 1 from *T* and so on up to all *k* chosen from set *T*. □

For example: Suppose
S = {a,b} with |S| = *m* = 2, and
T = {c,d,e} with |T| = *n* = 3 and
k = 2. So the sum on the right would be:

$$\sum_{i=0}^{2} \binom{2}{2-i} \binom{3}{i} = \binom{2}{2}\binom{3}{0} + \binom{2}{1}\binom{3}{1} + \binom{2}{0}\binom{3}{2}$$

Theorem:

$$\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}$$

for all natural numbers n ≥ 1.

Proof: Using lemma 1 we can write $\binom{n}{i}^2 = \binom{n}{i}\binom{n}{n-i}$.

Now we observe that the sum is just a special case of lemma 2, where *m* = *n*, and *k* = *n*, as follows:

□

$$\sum_{i=0}^{n} \binom{n}{n-i}\binom{n}{i} = \binom{n+n}{n}$$

Fibonacci Numbers
(See Chapter 4 of Discrete Mathematics: Elementary and Beyond)

Leonardo Fibonacci (c. 1170 – c. 1250) was in Italian mathematician.

The Fibonacci numbers are defined by a recursively defined sequence that occurs frequently in nature. Mathematical historians agree that this sequence was known in India well before Fibonacci.

The Fibonacci sequence is characterized by the following recursive function

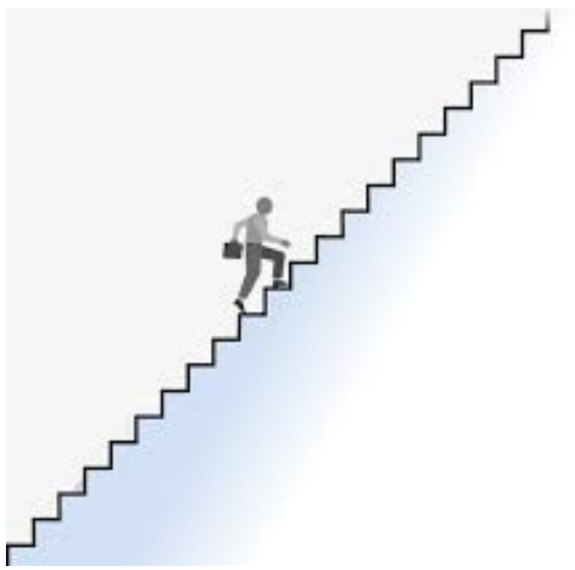
$F(1) = 1$, $F(2) = 1$
$F(n) = F(n-1) + F(n-2)$ for $n \geq 3$.

(Note: an equivalent definition starting at F(0) is
$F(0) = 0$ $F(1) = 1$
$F(n) = F(n-1) + F(n-2)$ *for* $n \geq 2$.*)*

The first few values of the Fibonacci sequence are
1, 1, 2, 3, 5, 8, 13, 21, …

Consider the following counting problem.

A staircase has n steps. You walk up the stairs either one step or two steps at a time. How many ways are there to reach the the n^{th} step?



Lets examine small values of n.

1 step 1 way.
2 steps 2 ways.
3 steps (1 + 1 + 1, 1 + 2, 2 + 1) 3 ways
4 steps ?

For 4 steps there are 5 ways.
(1 + 1 + 1 + 1, 1 + 1 + 2, 1 + 2 + 1, 2 + 1 + 1, 2 + 2)

Suppose J(n) is a function that gives the answer to this counting problem.

Observe that to count the number of ways of getting to step n we can focus on the last climbing step. It could either be a 1 step (count the number of ways to get to step n-1) or a 2 step (count the number of ways to get to step n-2)

So we have the following recursive function
$J(1) = 1$
$J(2) = 2$
$J(n) = J(n-1) + J(n-2)$ for $n \geq 3$.

The first few numbers in this sequence is:

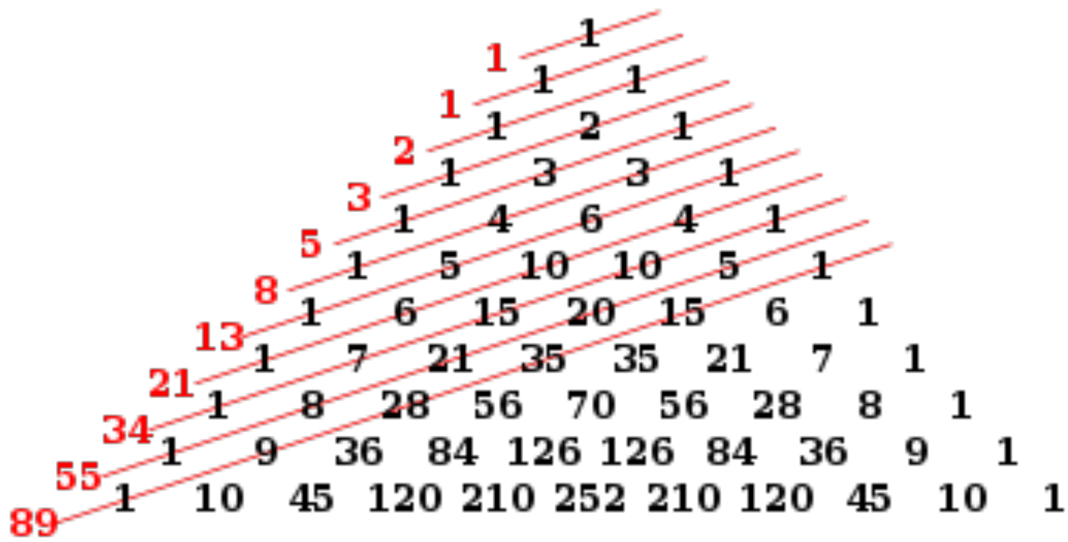
1, 2, 3, 5, 8, 13, 21, …

And it's easy to see that $J(n) = F(n+1)$ for $n \geq 1$.

Consider the following two counting problems:

We have n dollars to spend. Every day we either spend 1 dollar for a pop or 2 dollars for a bag of chips. In how many ways can we spend our money?

How many subsets are there of the set {1, 2, 3,…, n} that contain no two consecutive numbers?



Fibonacci numbers can be obtained by adding up entries in Pascal's triangle.

$$F(n) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k}$$

Where $\left\lfloor \dfrac{n-1}{2} \right\rfloor$ denotes $\dfrac{n-1}{2}$ rounded down.

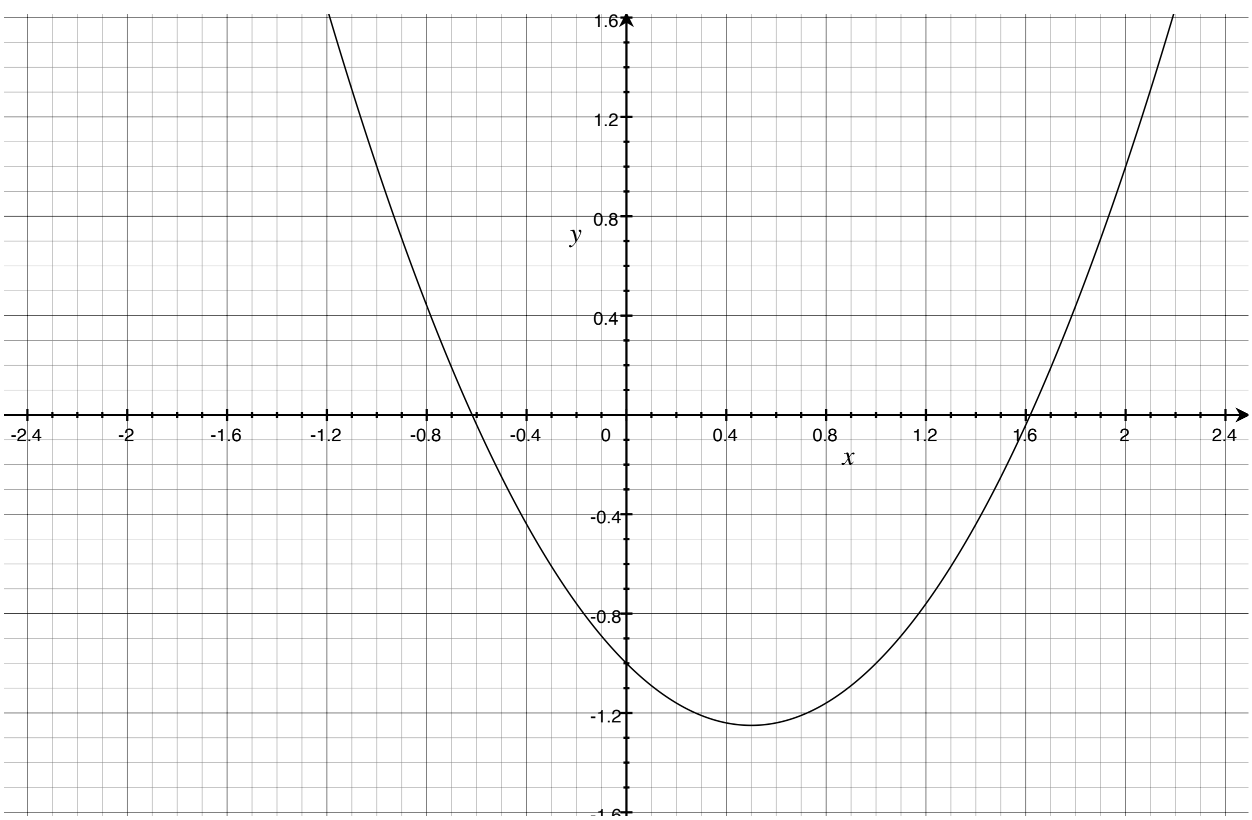
At this point you may be wondering what is the value of F(n).

Here is the expression, but it's not pretty.

$$F(n) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

Consider the equation

$$x^2 - x - 1 = 0$$



Let $\varphi = \dfrac{1+\sqrt{5}}{2} \approx 1.618$ and $\psi = \dfrac{1-\sqrt{5}}{2} \approx -0.618$.

So F(n) = $\dfrac{\varphi^n - \psi^n}{\sqrt{5}}$.

Also observe that:

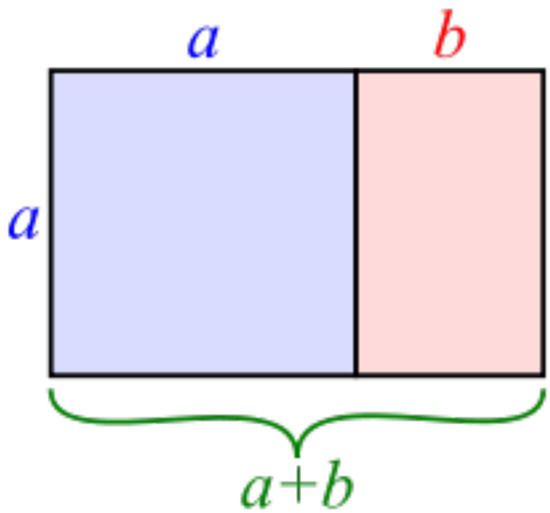
$\varphi^2 - \varphi - 1 = 0$ and $\psi^2 - \psi - 1 = 0$

And also that:

$$\varphi^n - \varphi^{n-1} - \varphi^{n-2} = 0, \text{ for } n \geq 2,$$

and

$$\psi^n - \psi^{n-1} - \psi^{n-2} = 0, \text{ for n} \geq 2.$$



The Golden Ratio

Given lengths *a* and *b* we say that they are in the golden ratio if:

$$\frac{a+b}{a} = \frac{a}{b} = \varphi$$

We could prove that:

$$F(n) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

By using mathematical induction. We will do something that requires less algebra.

Upper bound.

$F(n) \leq 2^{n-1}$, for all natural numbers n.

Base: $F(1) = 1 \leq 2^0$, $F(2) = 1 \leq 2^1$.
Induction Hypothesis: $F(j) \leq 2^{j-1}$ for j , $1 \leq j \leq k$.
Induction Step:

$$\begin{aligned} F(k+1) &= F(k) + F(k-1) \\ &\leq 2^{k-1} + 2^{k-2} \\ &\leq 2^k \qquad \square \end{aligned}$$

Lower bound.

$F(n) \geq \left(\frac{3}{2}\right)^{n-2}$, for all natural numbers n.

Base: $F(1) = 1 \geq \left(\frac{3}{2}\right)^{-1}$, $F(2) = 1 \geq \left(\frac{3}{2}\right)^{0}$.

Induction Hypothesis: $F(j) \geq \left(\frac{3}{2}\right)^{j-2}$ for j , $1 \leq j \leq k$.

Induction Step:

F(k+1) = F(k) + F(k-1)

$$\geq \left(\frac{3}{2}\right)^{k-2} + \left(\frac{3}{2}\right)^{k-3}$$

$$= \frac{3^{k-2}}{2^{k-2}} + \frac{3^{k-3}}{2^{k-3}} = \frac{3^{k-2}}{2^{k-2}} + \frac{2(3^{k-3})}{2^{k-2}}$$

$$= \frac{3^{k-3}(3+2)}{2^{k-2}} = \frac{(3^{k-3})}{2^{k-3}} \times \frac{5}{2}$$

$$\geq \frac{(3^{k-3})}{2^{k-3}} \times \frac{9}{4} = \left(\frac{3}{2}\right)^{k-3} \times \left(\frac{3}{2}\right)^{2}$$

$$= \left(\frac{3}{2}\right)^{k-1} \qquad \square$$

There are many interesting properties of the Fibonacci numbers. As a first example consider summing them.

We get

$1 = 1$

$1 + 1 = 2$

$1 + 1 + 2 = 4$

$1 + 1 + 2 + 3 = 7$

$1 + 1 + 2 + 3 + 5 = 12$

$1 + 1 + 2 + 3 + 5 + 8 = 20$

$1 + 1 + 2 + 3 + 5 + 8 + 13 = 33$

Can you see a pattern?

3, 5, 8 , 13, 21, 34 vs. 2, 4, 7, 12, 20, 33

Observe that F(1) + F(2) = F(4) -1,
and F(1) + F(2) + F(3) = F(5) -1,
and F(1) + F(2) + F(3) + F(4) = F(6) -1,
and F(1) + F(2) + F(3) + F(4) + F(5) = F(7) -1,
and F(1) + F(2) + F(3) + F(4) + F(5) + F(6) = F(8) -1, and
F(1) + F(2) + F(3) + F(4) + F(5) + F(6) + F(7) = F(9) -1.

Therefore we guess that:

$\sum_{i=1}^{n} F(i) = F(n+2) - 1$, for natural numbers $n \geq 2$.

And we can prove this by induction.

Base: F(1) + F(2) = F(4) -1

Induction Hypothesis: $\sum_{i=1}^{k} F(i) = F(k+2) - 1$

Induction Step:

$$\sum_{i=1}^{k+1} F(i) = \sum_{i=1}^{k} F(i) + F(k+1)$$

$$= F(k+2) - 1 + F(k+1)$$

$$= F(k+3) - 1 \qquad \square$$

Consider the sum of the first n even Fibonacci numbers. For reference the first few values of the Fibonacci sequence are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, …

Working out the first few terms we get

F(2) = 1
F(2) + F(4) = 4
F(2) + F(4) + F(6) = 12
F(2) + F(4) + F(6) + F(8) = 33

It appears that the sum of the first n even Fibonacci numbers is equal to the n+1st odd Fibonacci number minus 1, as shown below.
F(2) = 1 = F(3) - 1
F(2) + F(4) = 4 = F(5) -1
F(2) + F(4) + F(6) = 12 = F(7) -1
F(2) + F(4) + F(6) + F(8) = 33 = F(9) - 1

We can verify this guess using mathematical induction.

$$\sum_{i=1}^{n} F(2i) = F(2n+1) - 1, \text{ for all natural numbers } n.$$

Proof:

Base: F(2) = 1 = F(3) - 1.

Induction Hypothesis: $\sum_{i=1}^{k} F(2i) = F(2k+1) - 1$**.**

Induction Step:

$$\sum_{i=1}^{k+1} F(2i) = \sum_{i=1}^{k} F(2i) + F(2k+2)$$

$$= F(2k+1) - 1 + F(2k+2)$$

$$= F(2k+3) - 1 \quad \square$$