CISC 868 Fall 2011
Week 4

October 3, 2011

A taste of 3D

We looked at a few pictures illustrating the challenges encountered when attempting to solve problems in 3D. Some of the facts that were touched upon are:

- Porlyhedra is 3D may or may not be partitionable into tetrahedra, (triangulated). If the polyhedron is convex that triangulation is aways possible. Otherwise it is NP-complete to decide whether a non-convex polyhedron can be triangulated.
- The iterative algorithm for computing the convex hull of a set of points in 2D has its natural analog in 3D. However, the computational complexity explodes to $O(n^2)$ for the 3D version of the algorithm.

The Expected Number of Hull Points

Suppose that the input is a set of n points in 2D uniformly distributed in a unit square. (Think of picking n points from the unit square at random where each point has an equal probability of being picked.) A homework question asked to show that the expected number of points on the convex hull of a such a point set is $O(\log n)$.

A preliminary attempt at solving this problem showed that $O(\sqrt{n})$) is an upper bound. The argument is based on partitioning the square into n grid squares, each of area $1/n$.

To obtain the desired bound of $O(\log n)$ the notion of maximal points was defined. Maximal points come in four categories that can be identified as { ++, -+, –, +-} and corresponding to the sign of coordinates in the four quadrants of a Cartesian plane. It is not too hard to prove that the set of maximal points of all four categories is a superset of the hull points. Therefore, an upper bound on the expected number of maximal points is also an upper bound for the expected number of hull points.

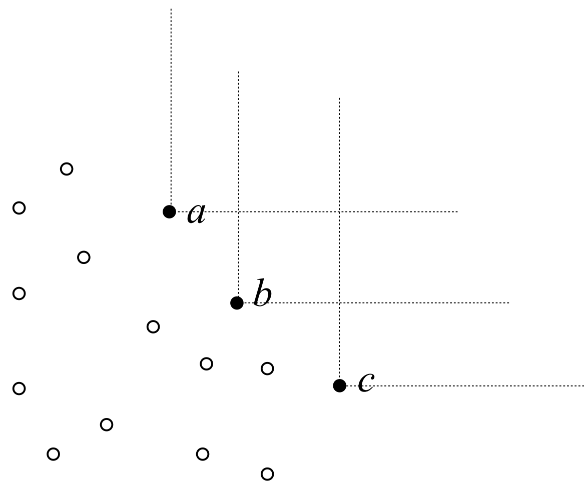

Figure 1: Points labelled a, b, and c are maximal. Observe that a and c are also hull points but b is not.

Definition 1. *Let p and q be points from a set S. We say that q ++* dominates *p if $x(p) < x(q)$ and $y(p) < y(q)$. A point $p \in S$ is ++* maximal *if there is no $q \in S$ that ++ dominates p.*

Assume that the points in S are labelled so that

$$x(s_0) < x(s_2) < \cdots < x(s_{n-1})$$

Furthermore, let κ denote the number of ++ maximal points. Thus we can express the expected value of κ, as:

$$E(\kappa) = \sum \text{prob}(s_i)\text{is } ++ \text{ maximal}$$

Then we can show that the probability that s_i is ++ maximal is $\frac{1}{n-i}$, because s_i only has to have a greater y cordinate that any point to its right to be maximal. This leads us to say that the expected number of ++ maximal points, that is, $E(\kappa)$, is given by the sum

$$E(\kappa) = \sum_{i=0}^{n-1} \frac{1}{n-i} = \sum_{i=1}^{n} \frac{1}{i}$$

This quantity is the *nth* Harmonic number denoted by H_n and it's value is in $\mathrm{O}(\log n)$.