The Voronoi diagram partitions the plane into cells, one per site.
Each cell consists of a locus of points that is closer to its site than to any other site.

The site closest to the star is easy to determine.

Each cell consists of a locus of points that is closer to its site than to any other site.
Every Voronoi vertex is equidistant to three sites, and the centre of an “empty” circle.

The dual of the Voronoi diagram of points in general position is a triangulation.
This triangulation is called the Delaunay triangulation.

Circumcircles of Delaunay triangles also possess the empty circle property.
Circumcircles of Delaunay triangles also possess the empty circle property

**Facts**

- Voronoi diagrams: Russian G. Voronoi (1907)
- Delaunay graph: Russian B. Delaunay (1934)
- $O(n \log n)$ worst case algorithms for constructing these structures in the plane.
- Structures generalize to higher dimensions and have been studied extensively.
Exactly one of $ab$ and $cd$ are locally Delaunay
Flipping

Exactly one of $ab$ and $cd$ are locally Delaunay

Flipping

Exactly one of $ab$ and $cd$ are locally Delaunay
Flipping

Exactly one of \( ab \) and \( cd \) are locally Delaunay

Flipping

When 4 or more points are cocircular, use the diagonal that is lexicographically smallest.
Angle 2 = 1/2 length of minor arc subtended by the edge $pr$

angle 3 < angle 2 < angle 1
The basic issue in the design of the algorithm is how to update the triangulation when a new site is added. In order to do this, we first investigate the basic properties of a Delaunay triangulation. Recall that a graphically maximum triangulation, and this triangulation must satisfy the empty circle condition, and hence is only possible when the quadrilateral is convex. Suppose that the initial triangle pair violates the empty circle condition, in that point $\triangle abc$ contains it in the current triangulation. In this case, we will need to argue that the expected number of times that a site is rebucketed is $O(n)$. As with any randomized incremental algorithm, the idea is to insert sites in random order, one at a time, and update the triangulation with each new addition. The issues involved with the analysis will be showing that after swapping, these other two angles cannot be smaller than the minimum of $\phi_{ab}$, $\phi_{bc}$, $\phi_{cd}$, and $\phi_{da}$. (Can you spot them?) It is not hard to show that, \[ \phi_{ab} > \theta_{ab} \quad \phi_{bc} > \theta_{bc} \quad \phi_{cd} > \theta_{cd} \quad \phi_{da} > \theta_{da}. \]