Compatible Geometric Matchings[☆]

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Abstract

This paper studies non-crossing geometric perfect matchings. Two such perfect matchings are *compatible* if they have the same vertex set and their union is also non-crossing. Our first result states that for any two perfect matchings M and M' of the same set of n points, for some $k \in \mathcal{O}(\log n)$, there is a sequence of perfect matchings $M = M_0, M_1, \ldots, M_k = M'$, such that each M_i is compatible with M_{i+1}. This improves the previous best bound of $k \leq n - 2$. We then study the conjecture: *every perfect matching with an even number of edges has an edge-disjoint compatible perfect matching.* We introduce a sequence of stronger conjectures that imply this conjecture, and prove the strongest of these conjectures in the case of perfect matchings that consist of vertical and horizontal segments. Finally, we prove that every perfect matching with n edges has an edge-disjoint compatible matching with approximately $4n/5$ edges.

Key words: geometric graph, compatible matching, convex-hull-connected segments, convexly independent segments, segments in convex position

1. Introduction

A *geometric graph* is a simple graph G, where the vertex-set $V(G)$ is a finite set of points in the plane, and each edge in $E(G)$ is a closed straight-line segment whose endpoints belong to $V(G)$. Throughout this paper, we assume that $V(G)$ is in general position; that is, no three vertices are collinear. A geometric graph is *non-crossing* if no two edges cross. That is, two edges may intersect only at a common endpoint. Two non-crossing geometric graphs are *compatible* if they have the same vertex set and their union is non-crossing.

In this paper, a *matching* is a non-crossing geometric graph in which every vertex has degree at most one. A matching is *perfect* if every vertex has degree exactly one. We say that a (perfect) matching is a (*perfect*) *matching of* its vertex set. Our focus is on compatible perfect matchings.

We first consider the problem of transforming a given perfect matching into another given perfect matching on the same vertex set. Let S be a set of n points in the plane, with n even. For perfect matchings M and M' of S, a *transformation between M and M' of length k* is a sequence

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$M = M_0, M_1, \ldots, M_k = M'$ of perfect matchings of S, such that M_i is compatible with M_{i+1}, for all $i \in \{0, 1, \ldots, k-1\}$. Houle et al. [15] proved that there is a transformation of length $n-2$ between any given pair of perfect matchings of S. The first contribution of this paper is to improve this bound from $n-2$ to $\mathcal{O}(\log n)$. This result is proved in Section 3. This upper bound is almost optimal, since Razen [24] recently proved an $\Omega(\log n / \log \log n)$ lower bound for this problem.

The remainder of the paper is concerned with the following conjecture. Two geometric graphs are *disjoint* if they have no edge in common. A matching is *even* or *odd* if the number of edges is even or odd.

Compatible Matching Conjecture. For every even perfect matching M, there is a perfect matching that is disjoint and compatible with M.

Note that the assumption that the given perfect matching is even is necessary, since there are odd perfect matchings that have no disjoint compatible perfect matching, as described in Section 4.

Section 5 describes progress toward the proof of this conjecture. In particular, we introduce a sequence of stronger conjectures that imply the Compatible Matching Conjecture.

In the next two sections we establish the Compatible Matching Conjecture for the following special cases: perfect matchings that consist of vertical and horizontal segments (Section 6), and perfect matchings that arise from convex-hull-connected sets of segments (Section 7).

In the final two sections we consider two relaxations of the Compatible Matching Conjecture. First we relax the requirement that the matching is perfect, and we prove that every perfect matching with n edges has a disjoint compatible (partial) matching with approximately $4n/5$ edges (Section 8). Finally, we prove a weakened version of the Compatible Matching Conjecture in which certain types of crossings are allowed (Section 9).

1.1. Related Work

Instead of transforming perfect matchings, Aichholzer et al. [3] considered transforming spanning trees of a fixed set of n points, and established the following results. Start with any non-crossing spanning tree T, and let $f(T)$ be the shortest spanning tree that does not cross T. Then $f(T)$ is non-crossing. In addition, iterating the operator f must stop at some point, because the total length of the edges is decreasing. Aichholzer et al. [3] proved that this process always leads to a minimum spanning tree, for every starting

tree T. Moreover, it takes $\mathcal{O}(\log n)$ steps to reach a minimum spanning tree, and for some starting trees, $\Omega(\log n)$ steps are required. As a corollary, there is a transformation of length $\mathcal{O}(\log n)$ between any two spanning trees. Whether this bound is tight is of some interest.

Partially motivated by connections with pseudo-triangulations, Aichholzer et al. [2] conjectured that there is a transformation of length $o(\log n)$ between any two spanning trees. Recently Buchin et al. [9] proved an $\Omega(\log n / \log \log n)$ lower bound for this question.

There is another problem that has attracted substantial research and is closely related to the Compatible Matching Conjecture. In general, given a set S of n pairwise disjoint segments it is not always possible to form a polygon with $2n$ sides such that every second segment on its boundary belongs to S (an *alternating* polygon). Toussaint raised the computational problem of deciding whether an alternating polygon exists, which was extensively studied by Rappaport and other authors [22, 23]. Later Mirzaian [16] conjectured that there is a polygon such that every segment from S is a side or an internal diagonal (a *circumscribing polygon*); this was disproved by Urabe and Watanabe [28]. Pach and Rivera-Campo [19] proved that there is a circumscribing polygon of size $\Omega(n^{1/3})$ (although this cycle may cross other segments). Mirzaian [16] also conjectured that there is a polygon such that every segment from S is a side, an internal diagonal, or an external diagonal. This conjecture was finally proved by Hoffmann and Tóth [14]. In related work, Bose et al. [8] proved that S can be augmented by new segments to form a spanning tree with maximum degree 3, and Hoffmann and Tóth [13] and Tóth [25] studied the maximum length of an alternating path compatible with S.

The Compatible Matching Conjecture follows the lines of the original formulation of the preceding problem, since it is equivalent to the conjecture that there is a *set* of pairwise disjoint simple polygons, with a total of $2n$ edges, such that every segment from S lies on the boundary of one of them.

2. Tools

2.1. Matchings in a Polygon

The following result by Abellanas et al. [1] is used repeatedly throughout the paper.

Lemma 1 ([1]). *Let P be a simple polygon, let R be the set of reflex vertices of P, and let S be any finite set of points on the boundary of P or in its interior, such that $R \subseteq S$ and $|S|$ is even. Then there is a perfect matching M of S such that every segment in M is inside the (closed) polygon P.*

While in general, the Compatible Matching Conjecture is false for odd perfect matchings of point sets in convex position, the following lemma provides an important special case when a disjoint compatible perfect matching always exists.

Lemma 2. *Let P be a set of points in convex position. Let M be a matching of P such that every segment in M is on the boundary of the convex hull of P. Then there is a perfect matching of P that is disjoint and compatible with M if and only if $|P|$ is even and if $|P| = 2$ then $E(M) = \emptyset$.*

Proof. The necessity of the conditions are obvious. We prove the sufficiency by induction on $|P|$. The base cases with $|P| \leq 4$ are easily verified. Now suppose that $|P| \geq 6$ is even. Thus there are consecutive vertices v and w in P that are not adjacent in M. Let $P' := P - \{v, w\}$. Let M' be the subgraph of M induced by P'. Thus M' is a matching of P' such that every segment in M' is on the convex hull of P'. Since $|P'| \geq 4$ is even, by induction, M' has a disjoint compatible perfect matching M''. Let M''' be the geometric graph obtained from M'' by adding the vertices v and w, and adding the edge vw. Now M''' is non-crossing, since v and w are consecutive on the convex hull of P. Since $vw \notin E(M)$, M and M''' are disjoint. □

2.2. Segment Extensions

Let M be a perfect matching, and let C be a (possibly unbounded) convex set in the plane, such that every segment in M that intersects C has at least one endpoint in C. Let M_1 be the set of segments in M with exactly one endpoint in C. Let M_2 be the set of segments in M with both endpoints in C. We ignore the segments in M outside of C.

As illustrated in Figure 1, an *extension* of M in C is a set of segments and rays obtained as follows. For each segment $s \in M_1 \cup M_2$ in some given order, extend s by a ray, in both directions if $s \in M_2$, and in the direction into C if $s \in M_1$. Each ray is extended until it hits another segment, the boundary of C, or a previous extension, or the ray *goes to infinity* if it is not blocked. An extension L of M defines a convex subdivision of C with $|M_1| + |M_2| + 1$ cells, since the extension of each segment splits one cell into two cells. Since

the vertices of M are in general position by assumption, each vertex of M that is in C is on the boundary of exactly two cells of the convex subdivision.

The *dual multigraph G* associated with M and L is the (non-geometric) multigraph whose vertices are the cells of this convex subdivision. For every vertex v of M that is in C, add an edge to G between the vertices that correspond to the two cells of the convex subdivision of which v is on the boundary. Thus G has $|M_1| + |M_2| + 1$ vertices and $|M_1| + 2|M_2|$ edges. One can show that G is always connected.

As illustrated in Figure 1(c), consider G to be drawn in the plane with each vertex of G in the interior of the corresponding cell of the convex subdivision formed by L. Each edge xy of G corresponding to a vertex v of M is drawn as a simple curve connecting x and y through v. Clearly G can be drawn in this way without edge crossings. Moreover, if an edge of G crosses L then it crosses at a vertex of M.

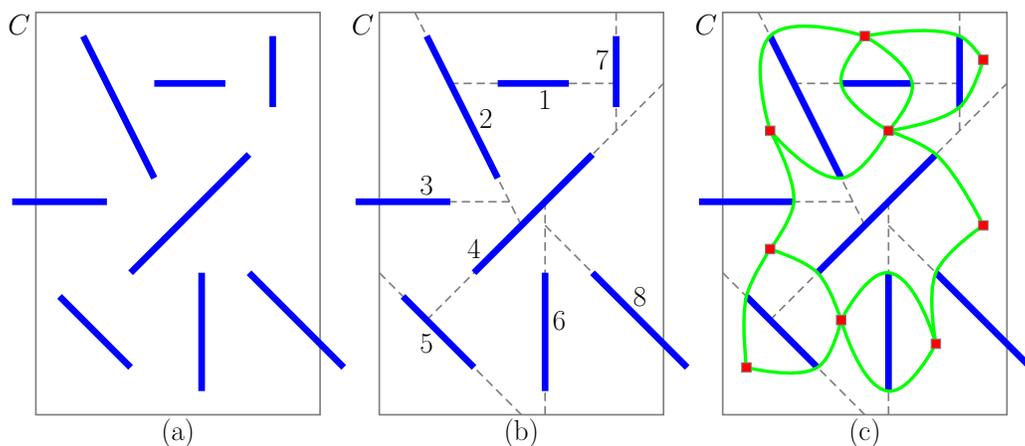The above properties of extensions of segments are folklore; see [18].

Figure 1: (a) A given perfect matching M and convex region C. (b) The extension of M in the order shown produces a subdivision of C. (c) The associated dual multigraph G.

2.3. Even Orientations

Our third tool is non-geometric. A *multigraph* allows parallel edges. A multigraph is *even* if it has an even number of edges, and *odd* otherwise. An orientation of a multigraph is *even* if every vertex has even indegree. Frank et al. [10] and Frank and Király [11] characterised multigraphs that admit an even orientation.

Lemma 3 ([10, 11]). *A multigraph admits an even orientation if and only if every component is even.*

It is easily seen that a tree T with an even number of edges has a unique even orientation.

3. Transforming Matchings

In this section we prove the following theorem.

Theorem 4. *For every set S of $2n$ points in general position, there is a transformation of length at most $2\lceil \log_2 n \rceil$ between any given pair of perfect matchings of S.*

We begin with some preliminary lemmas.

Lemma 5. *Let M be a perfect matching. Let t be a line cutting an even number of segments in M, but containing no vertex of M. Let H be a half-plane determined by t. Let S be the set of vertices of M that are in H. Then there is a perfect matching M' of S such that $M \cup M'$ is non-crossing.*

First Proof. Say m segments of M are cut by t, and n segments of M are contained in H. As described in Section 2.2, consider an extension of M in H. The obtained subdivision of H has $m + n + 1$ convex cells, and the dual multigraph G is connected. Since m is even, the number of edges of G, $m + 2n$, is also even. By Lemma 3, G admits an even orientation. For each oriented edge xy of G corresponding to a vertex v in S, *assign v* to the cell corresponding to y. Thus each cell C is assigned an even number of vertices. Let S_C be the set of vertices assigned to cell C. Since C is convex, there is a perfect matching of S_C that is compatible with the matching of S_C induced by M. (We cannot conclude that these matchings are disjoint, as in Lemma 2, since it is possible that $|S_C| = 2$ and the two points are endpoints of the same segment.) The union of these matchings, taken over all the convex cells C, is a perfect matching M' of S, such that $M \cup M'$ is non-crossing. □

Second Proof. Without loss of generality, t is horizontal, and no segment in M is vertical. Let C be a rectangle containing S whose base side is contained in t. For each segment vw of M with at least one endpoint in S, let x be a point infinitesimally below the interior of vw, such that x is not vertically

above a vertex in S. Now, thicken vw into the triangle vxw. Moreover, if both v and w are in S, then draw an infinitesimally wide axis-parallel rectangle from x downward until it reaches t or another segment of M. As illustrated in Figure 2, removing the infinitesimal elements from C and possibly removing portions of C bordering t that do not intersect M, we obtain a simple polygon P whose reflex vertices are precisely the vertices in S. By Lemma 1 with $R = S$, there is a perfect matching M' of S, such that every segment in M' is inside P. Thus $M \cup M'$ is non-crossing. □
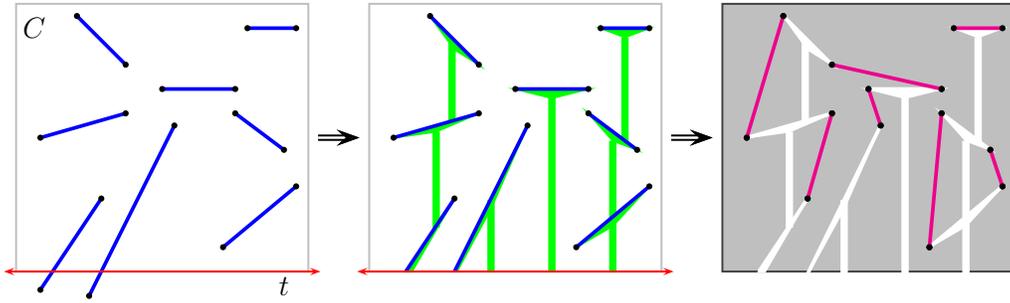

Figure 2: Second proof of Lemma 5.

Lemma 6. *Let M be a perfect matching. Let t be a line cutting an even number of segments of M, but containing no vertex of M. Let S_1 and S_2 be the sets of vertices of M lying in the two halfplanes determined by t. Then there is a perfect matching M_1 of S_1 and a perfect matching M_2 of S_2, such that M and $M_1 \cup M_2$ are compatible (but not necessarily disjoint).*

Proof. Apply Lemma 5 to each half-plane determined by t, to obtain a perfect matching M_1 of S_1 and a perfect matching M_2 of S_2, such that $M \cup M_1$ is non-crossing and $M \cup M_2$ is non-crossing. Now $M_1 \cup M_2$ is non-crossing since M_1 and M_2 are separated by t. Hence $M_1 \cup M_2$ is a perfect matching of the vertex set of M, and it is compatible with M. □

Let S be a set of $2n$ points in general position in the plane. Without loss of generality, no two points in S have the same X-coordinate. Order the points $p_1, p_2, \ldots, p_{2n}$ in increasing order of their X-coordinates. Let $N(S)$ be the *canonical* perfect matching that links p_{2i-1} with p_{2i} for each $i \in \{1, 2, \ldots, n\}$.

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Lemma 7. *For every set S of $2n$ points in general position, and for every perfect matching M of S, there is a transformation between M and $N(S)$ of length $\lceil \log_2 n \rceil$.*

Proof. Let $k(n) = \lceil \log_2 n \rceil$. We proceed by induction on n. With $n = 1$, every perfect matching of S is canonical, and we are done since $k(1) = 0$. Now assume that $n > 1$ and the lemma holds for all values less than n. Let t be a vertical line with $2\lfloor n/2 \rfloor$ points of S to the left of t, and $2\lceil n/2 \rceil$ points of S to the right of t. Let S^ℓ and S^r be the sets of points in S respectively to the left and right of t. Say t cuts m edges of M. The $2\lfloor n/2 \rfloor - m$ points of S^ℓ that are incident to an edge of M not cut by t are matched by M. Thus m is even. By Lemma 6, there is a perfect matching M^ℓ of S^ℓ and a perfect matching M^r of S^r, such that M and $M^\ell \cup M^r$ are compatible. Now apply induction to M^ℓ and M^r. Observe that $k(\lfloor n/2 \rfloor) \leq k(\lceil n/2 \rceil) \leq k(n) - 1$. Thus there are transformations

$$M^\ell = M_0^\ell, M_1^\ell, \ldots, M_{k(n)-1}^\ell = N(S^\ell) \text{ and}$$
$$M^r = M_0^r, M_1^r, \ldots, M_{k(n)-1}^r = N(S^r),$$

where each M_i^ℓ is compatible with M_{i+1}^ℓ, and each M_i^r is compatible with M_{i+1}^r. Let $M_i := M_{i-1}^\ell \cup M_{i-1}^r$ for each $i \in \{1, 2, \ldots, k(n)\}$. Since M_{i-1}^ℓ and M_{i-1}^r are separated by t, M_i is a perfect matching of S, and M_i is compatible with M_{i+1} for each $i \in \{1, 2, \ldots, k(n) - 1\}$. By Lemma 6, M and $M_1 = M^\ell \cup M^r$ are compatible. Since $N(S) = N(S^\ell) \cup N(S^r) = M_{k(n)}$,

$$M, M_1, \ldots, M_{k(n)}$$

is a transformation between M and $N(S)$ of length $k(n)$. $\qquad\square$

Proof of Theorem 4. For perfect matchings M and M' of S, by Lemma 7, there are transformations

$$M = M_0, M_1, \ldots, M_{k(n)} = N(S) \text{ and}$$
$$M' = M_0', M_1', \ldots, M_{k(n)}' = N(S).$$

Thus $M = M_0, M_1, \ldots, M_{k(n)}, M_{k(n)-1}', M_{k(n)-2}' \ldots, M_0' = M'$ is a transformation between M and M' of length $2k(n)$. $\qquad\square$

4. Odd Matchings

In the remainder of the paper we study the Disjoint Compatible Matching conjecture. First, in this section, we show why this conjecture is false for odd perfect matchings. That is, we describe classes of odd perfect matchings that have no disjoint compatible perfect matching. An odd number of parallel chords of a circle form such a matching, as illustrated in Figure 3.
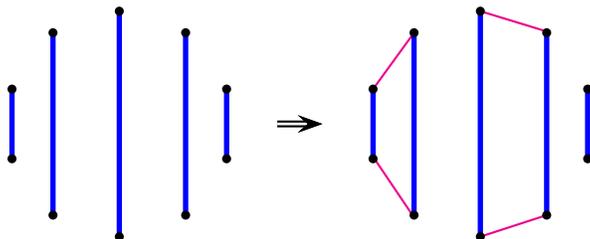

Figure 3: A 5-edge perfect matching with no disjoint compatible perfect matching.

For a more general example, start with a perfect matching of n black segments enclosed in a bounded convex region C. In some arbitrary order, extend each black segment, one by one, until it hits the boundary of C, or stop an ϵ distance from another segment, or extension of a segment. This gives a new perfect matching with n blue segments. The blue segments form $n + 1$ 'regions' inside C. In the middle of each region insert a short red segment. The blue and red segments together form a perfect matching M with $2n + 1$ segments, as illustrated in Figure 4. No two red vertices are visible (for small enough ϵ). So if M has a disjoint compatible perfect matching, then every red vertex is paired with a blue vertex, which is impossible because there are $2n + 2$ red vertices and $2n$ blue vertices. Thus M has no disjoint compatible perfect matching. In fact, in the visibility graph of $V(M)$ minus $E(M)$, the red vertices form an independent set with more than half the vertices. Hence the visibility graph of $V(M)$ minus $E(M)$ has no (graph-theoretic) perfect matching, which in turn implies that M has no disjoint compatible perfect matching.

5. Compatible Disjoint Matchings

In this section we introduce a sequence of conjectures that imply the Compatible Matching Conjecture. Our general approach, given a perfect matching M, is to first compute an extension L of M, then assign each
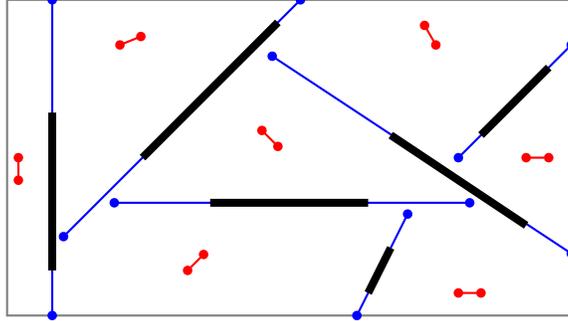
Figure 4: The blue and red segments form an odd perfect matching with no disjoint compatible perfect matching.

vertex of M to one of its two neighbouring cells in the convex subdivision formed by L, and then compute a perfect matching of the vertices assigned to each cell, the union of which is a perfect matching that is disjoint and compatible with M. The assignment of the vertices of M to cells is modelled by an orientation of the edges of the dual multigraph.

Extension Conjecture. Every even perfect matching M has an extension L, such that the associated dual multigraph G admits an even orientation, with the property that whenever a vertex v of G has indegree 2, the two incoming edges at v do not arise from the same segment in M.

Lemma 8. *The Extension Conjecture implies the Compatible Matching Conjecture.*

Proof. Given a perfect matching M, let L be an extension of M with the properties stated in the Extension Conjecture. Let G be the dual multigraph associated with M and L. For each oriented edge xy of G corresponding to a vertex v of M, *assign* v to the cell y. For each cell y, let S_y be the set of vertices assigned to y. Since the orientation of G is even, $|S_y|$ is even. Now y is convex, and every vertex in S_y is on the boundary of y. Thus S_y is in convex position. Let M_y be the subgraph of M induced by S_y. Since no segment in M intersects the interior of y, M_y is a matching of S_y, and every segment in M_y is on the convex hull of S_y. Moreover, by the final assumption in the Extension Conjecture, if $|S_y| = 2$, then the two vertices are not adjacent in M_y. Thus Lemma 2 is applicable to S_y with the matching M_y. Thus M_y has a disjoint compatible perfect matching M'_y. Let $M' := \cup_y M'_y$. Since each point is assigned to exactly one cell, M' is a perfect matching. Since

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the cells are disjoint, and each M'_y is non-crossing, M' is non-crossing. Since the interior of every edge in M'_y is contained in the interior of y, and every edge in M only intersects y at a vertex on the boundary, M and M' are compatible. □

Two Subgraphs Conjecture. Every even perfect matching M has an extension L, such that the associated dual multigraph G has an edge-partition into two subgraphs G_1 and G_2, such that each component of G_1 is even, each component of G_2 is even, and for every segment vw of M, the edge of G corresponding to v is in a different subgraph from the edge of G corresponding to w.

Lemma 9. *The Two Subgraphs Conjecture implies the Extension Conjecture.*

Proof. Given a perfect matching M, let L be an extension of M with the properties stated in the Two Subgraphs Conjecture. Since each component of G_1 and G_2 is even, by Lemma 3, each of G_1 and G_2 admit an even orientation. The union of the even orientations of G_1 and G_2 define an even orientation of G, such that if a vertex x of G has indegree 2, then the two incoming edges at x are both in G_1 or both in G_2, and thus arise from distinct segments in M. Hence the even orientation of G satisfies the requirements of the Extension Conjecture. □

Two Trees Conjecture. Every (even or odd) perfect matching M has an extension L, such that the associated dual multigraph G has an edge-partition into two trees, and for every segment vw of M, the edge of G corresponding to v is in a different tree from the edge of G corresponding to w.

Lemma 10. *The Two Trees Conjecture implies the Two Subgraphs Conjecture.*

Proof. Let M be a perfect matching with n edges. Assuming the Two Trees Conjecture, M has an extension L, such that the associated dual multigraph G has an edge-partition into two trees T_1 and T_2. Now G has $n+1$ vertices and $2n$ edges. Each tree has at most $n+1$ vertices, and thus has at most n edges. Since G has $2n$ edges, each tree has exactly n edges. That is, each tree is a spanning tree of G. In the case that M is even (which is assumed in the Two Subgraphs Conjecture), n is even. Thus T_1 and T_2 are connected subgraphs, each with an even number of edges, as desired. □

Some notes on the Two Trees Conjecture are in order:

- Tutte [27] and Nash-Williams [17] independently characterised the multi-graphs G that contain two edge-disjoint spanning trees as those that have at least $2k-2$ cross-edges in every partition of $V(G)$ into k parts. (A *cross-edge* has endpoints in distinct parts of the partition.)

- It is easily seen that in every minimum counterexample to the Two Trees Conjecture, for every segment r, there exists segments s and t, such that the line extending r intersects s, and the line extending t intersects r.

- Motivated by the present paper, Benbernou et al. [4] announced progress on the Two Trees Conjecture, but the proof was incorrect [5].

6. Vertical-Horizontal Matchings

Theorem 11. *Every perfect matching M consisting of vertical and horizontal segments satisfies the Two Trees Conjecture.*

Proof. Let L be an extension of M constructed as follows. First extend each horizontal segment in M in both directions until it hits some vertical segment or goes to infinity. Then extend each vertical segment in M in both directions until it hits some horizontal segment, an extension of some horizontal segment, or goes to infinity. Let G be the dual multigraph associated with M and L.

Colour the edges of G *red* and *green* as follows. For each horizontal segment vw in M, where v is the left endpoint and w is the right endpoint, colour the edge of G through v red, and colour the edge of G through w green. For each vertical segment vw in M, where v is the bottom endpoint and w is the top endpoint, colour the edge of G through v red, and colour the edge of G through w green, as illustrated in Figure 5.

We claim that both the red and green subgraphs of G are trees. Suppose on the contrary that G has a monochromatic cycle C. Since the construction is symmetric between green and red, without loss of generality, C is red. Now C is a simple closed curve drawn without crossings in the plane. Let D be the union of C with its interior.

First suppose that some horizontal segment of M intersects D. Let r be the topmost horizontal segment of M that intersects D. (r is uniquely

determined since the vertices of M are in general position.) Since the two edges of G that pass through the endpoints of r receive distinct colours, C does not intersect both endpoints of r. If some endpoint of r is in the interior of C, then let v be that endpoint (choosing v arbitrarily if both endpoints are in the interior). Otherwise, since an edge of G only intersects a segment of M at the endpoint of that segment, some endpoint v of r is on C. In both cases, the extension of r from v enters the interior of C and thus does not go to infinity because otherwise it would intersect C at some point other than an endpoint of r. The extension of r from v is not blocked by some vertical extension because the horizontal segments were extended before the vertical segments. Thus the extension of r from v is blocked by some vertical segment s, and s intersects D. Let w be the top endpoint of s. Thus the Y-coordinate of w is greater than that of v. Now C does not pass through w because the edge of G through w is green. The upward extension of s does not go to infinity because otherwise it would intersect C at some point other than an endpoint of s. Thus the upward extension of s is blocked by some horizontal segment t, and t intersects D. Thus t is a horizontal segment of M that intersects D and is higher than r. This contradiction proves that no horizontal segment of M intersects D.

Every edge of C passes through the endpoint of some segment s, in which case s intersects D. Thus some segment of M intersects D. Hence some vertical segment r of M intersects D. As in the previous case, if some endpoint of r is in the interior of C then let v be that endpoint. Otherwise, some endpoint v of r is on C. In both cases, the extension of r from v does not go to infinity because otherwise it would intersect C at some point other than an endpoint of r. Thus the extension of r from v is blocked by some horizontal segment s, and s intersects D, which is a contradiction.

Hence there is no monochromatic cycle in G. If M has n edges, then G has $n + 1$ vertices, the red subgraph has n edges, and the green subgraph has n edges. Every cycle-free graph with $n + 1$ vertices and n edges is a spanning tree. Thus the red subgraph is a tree and the green subgraph is a tree. By construction, for every segment vw of M, the edge of G passing through v is in a different tree from the edge of G passing through w. Thus the requirements of the Two Trees Conjecture is satisfied. □

Theorem 11 and Lemmas 8–10 imply:

Corollary 12. *Every even perfect matching consisting of vertical and horizontal segments has a disjoint compatible perfect matching.*
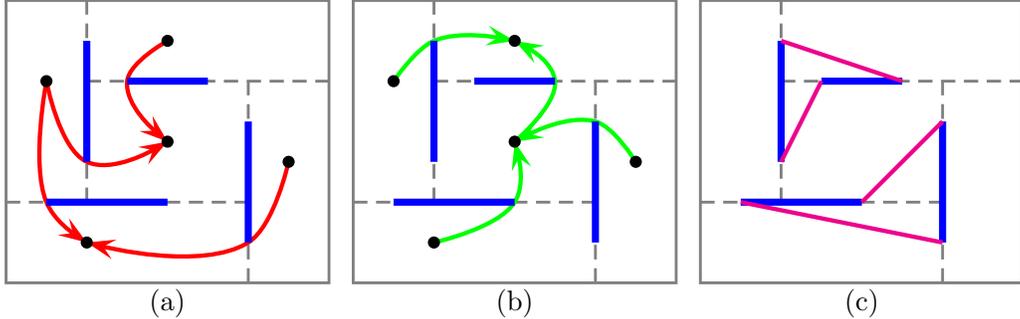
Figure 5: For a perfect matching of horizontal and vertical segments: (a) the evenly oriented red spanning tree of the dual multigraph, (b) the evenly oriented green spanning tree, and (c) the compatible disjoint perfect matching determined by our algorithm.

7. Convex-Hull-Connected Matchings

A set M of pairwise disjoint segments is *convex-hull-connected*[8] if each segment has at least one endpoint on the boundary of the convex hull of M. This restriction has appeared in the context of augmenting a set of segments to form a simple polygon. Rappaport et al. [23] gave an $\mathcal{O}(n \log n)$ time algorithm to determine whether a set of disjoint convex-hull-connected segments admits an alternating polygon. Moreover, Mirzaian [16] showed that every set of n disjoint convex-hull-connected segments admits a circumscribing polygon, which can be constructed in $\mathcal{O}(n \log n)$ time.

Theorem 13. *For every even convex-hull-connected perfect matching M, there is a perfect matching that is disjoint and compatible with M.*

Proof. We proceed by induction on the number of segments in M. A segment vw in M is a *splitter* if v and w are non-consecutive points on the boundary of the convex hull (amongst the set of endpoints of segments in M).

First suppose that there is a splitter segment vw in M. Of the sets of segments on the two sides of vw, one has an even non-zero number of segments, and the other has an odd number of segments. Group vw with the odd-sized set. Thus M is now partitioned into two even convex-hull-connected perfect matchings M_1 and M_2. By induction, there is a perfect matching M_1' that is disjoint and compatible with M_1, and there is a perfect

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[8]or *convexly independent* [7, 6] or *in convex position* [21, 20, 26]

matching M_2' that is disjoint and compatible with M_2. Hence $M_1' \cup M_2'$ is a perfect matching that is disjoint and compatible with M.

Now assume M has no splitter segment; refer to Figure 6. A *gap* is an edge of the convex hull of M that is not a segment in M. Since M is even and there are no splitter segments, the number of gaps is even. Let B be a set of alternate gaps on the convex hull. Thus B forms a set of segments, such that for every segment xy in M, exactly one of x and y is an endpoint of a segment in B. For each segment xy in M with exactly one endpoint, say x, on the convex hull, let $W(xy)$ be an infinitesimally thick wedge centred at y containing xy. Let P be the polygon obtained from the convex hull of M by removing each $W(xy)$. Thus every reflex vertex of P is an endpoint of a segment in M not intersecting B. Since M is even and B includes exactly one endpoint from each segment in M, the number of endpoints of segments in M not intersecting B is even. By Lemma 1, there is a perfect matching Q of the set of endpoints of segments in M not intersecting B, such that every segment in Q is inside polygon P. Since every segment in B is on the boundary of the convex hull, $B \cup Q$ is a perfect matching that is disjoint and compatible with M. □
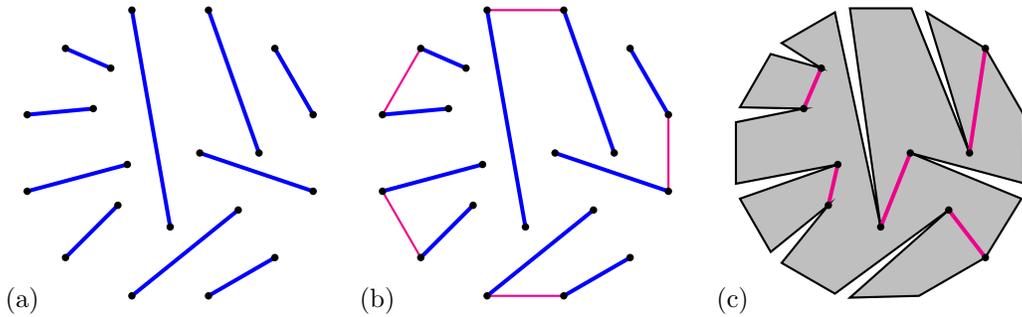

Figure 6: Proof of Theorem 13: (a) a convex-hull-connected set M of segments, (b) the set B of alternate gaps, (c) the polygon P and matching Q.

Note that the assumption that M is even is needed in Theorem 13—because of the instance in Figure 3 for example.

8. The 4/5 Theorem

Given that the Compatible Matching Conjecture has remained elusive, it is natural to consider how large a disjoint compatible matching can be

guaranteed.

Theorem 14. *Let S be a set of $2n$ points in the plane in general position, with n even, and let M be a perfect matching of S. Then there is a matching M' of S with at least $\frac{1}{5}(4n-1)$ segments, such that M and M' are compatible and disjoint.*

Proof. Without loss of generality, no segment is vertical. Fix a bounding box around the segments. First extend each segment to the right (in any order). Then extend each segment to the left (in any order). We obtain a convex subdivision with $n+1$ faces. Let G be the corresponding dual multigraph. So G has $n+1$ vertices. Colour each edge of G that corresponds to a right endpoint *red*. Colour each edge of G that corresponds to a left endpoint *blue*. Let R and B be the subgraphs of G, both with vertex set $V(G)$, respectively consisting of the red and blue edges. Each of B and R have $n+1$ vertices and n edges. This construction is well known [18]. Hoffmann et al. [12] proved that B is connected, and is thus a spanning tree of G.

For each odd component X of R, there is an edge e in X, such that $X-e$ has no odd component. (*Proof*: If X has a degree-1 vertex v, then let e be the edge incident to v. Otherwise X has a cycle, and let e be an arbitrary edge in the cycle.) Delete one such edge e from each odd component of R. We are left with a subgraph R' of R with no odd component. Since n is even, the one component of B is even. By construction, for every segment vw of M, the edge of G corresponding to v is coloured differently from the edge of G corresponding to w. Thus the technique used in the proofs of Lemmas 8 and 9 is applicable to the subgraphs of G induced by B and R'. We obtain a partial matching M' of S that is disjoint and compatible with M, and the number of segments in M' equals half the number of edges in $B \cup R'$, which is $2n$ minus the number of odd components in R. Lemma 15 below bounds the number of odd components in a planar graph. This result applied to R (which has $n+1$ vertices, n edges, and thus has some component not isomorphic to K_2) implies that R has at most $\frac{1}{5}(3(n+1)-n-1) = \frac{2}{5}(n+1)$ odd components. Hence M' has at least $\frac{1}{2}(2n - \frac{2}{5}(n+1)) = \frac{1}{5}(4n-1)$ segments. $\square$

Lemma 15. *Let $f(G)$ be the number of odd components in a graph G. Then every planar graph G with n vertices and m edges has $f(G) \leq \frac{1}{5}(3n - m)$, with equality if and only if every component of G is K_2.*

Proof. We proceed by induction on the number of components in G.

For the base case, suppose that G has one component. If $n = 1$ then $f(G) = 0 < \frac{3}{5} = \frac{1}{5}(3n - m)$. If $n = 2$ then $f(G) = 1 = \frac{1}{5}(3n - m)$. If $n \geq 3$ then $f(G) = 1 < \frac{6}{5} \leq \frac{1}{5}(3n - (3n - 6)) \leq \frac{1}{5}(3n - m)$. Now assume that G has at least two components.

Suppose that G has an isolated vertex v. By induction, $f(G) = f(G - v) \leq \frac{1}{5}(3(n-1) - m) < \frac{1}{5}(3n - m)$. Now assume that G has no isolated vertices.

Suppose that G has a component $H = K_2$. Then $f(G) = f(G - H) + 1 \leq \frac{1}{5}(3(n-2) - (m-1)) + 1 = \frac{1}{5}(3n - m)$. Moreover, suppose that $f(G) = \frac{1}{5}(3n - m)$. Then $f(G - H) = \frac{1}{5}(3(n-2) - (m-1))$, and by induction, every component of $G - H$ is K_2, which implies that every component of G is K_2. Now assume that G has no K_2 component.

Let X be a component of G with p vertices and q edges. By induction, $f(G) \leq 1 + f(G - X) \leq 1 + \frac{1}{5}(3(n-p) - (m-q)) = \frac{1}{5}(3n - m + q - 3p + 5) < \frac{1}{5}(3n - m)$ since $q \leq 3p - 6$. $\qquad\square$

We now show that the analysis of the algorithm in the proof of Theorem 14 is tight. First note that if an n-vertex m-edge planar graph G has one component that is maximal planar on an odd number of vertices, and every other component is K_2, then $f(G) = \frac{1}{5}(3n - m - 1)$. Figure 7 shows a set of segments such that by applying the algorithm in the proof of Theorem 14, the obtained graph R has one component that is maximal planar on an odd number of vertices, and every other component is K_2. It follows that for this set of segments, the algorithm in Theorem 14 will produce a matching with $\frac{1}{5}(4n - 1)$ segments.

9. Matchings with Crossings

Now we relax the Compatible Matching Conjecture by allowing crossings.

Theorem 16. *Let M be an even perfect matching with no vertical segment. Let S be the set of left endpoints of M, and let T be the set of right endpoints of M. Then there is a perfect matching M_S of S, and a perfect matching M_T of T, such that no edge in M crosses an edge in $M_S \cup M_T$ (but an edge in M_S might cross an edge in M_T).*

Proof. Let C be a convex polygon bounding M. In a similar fashion to the second proof of Lemma 5 and the proof of Theorem 13, extend each segment of M by an infinitesimally thickened ray from its left endpoint. Removing the thickened rays from the interior of C, we obtain a polygon whose reflex
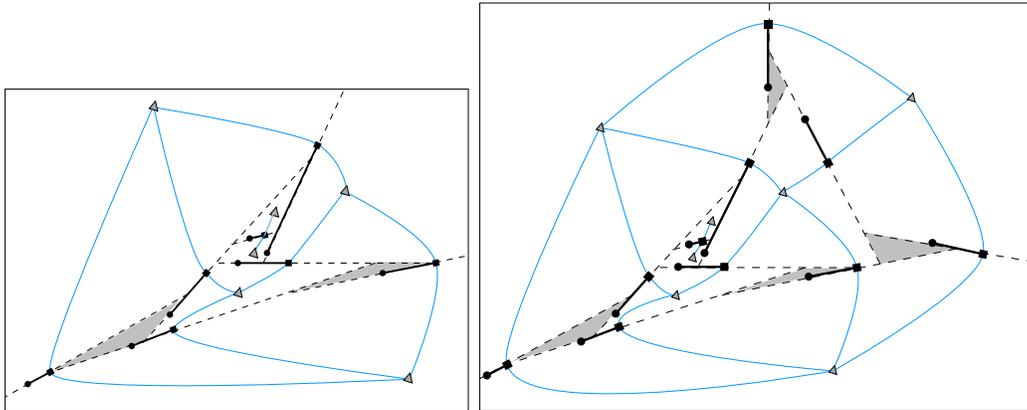
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Figure 7: Consider the seven segments in the figure on the left. It is possible to extend the segments (all right endpoints first), so that the graph R in Theorem 14 (the spanning subgraph of the dual multigraph consisting of the edges passing through the right endpoints) has two components, K_4 and K_2. Placing an additional segment in each of the two shaded regions creates two additional K_2 components in R. Then R has one component that is maximal planar, and every other component is K_2. By repeatedly adding two more segments as shown in the figure on the right, we obtain arbitrarily large sets of segments such that the graph R has one maximal planar component, and every other component is K_2.

vertices are the right endpoints of the segments in M. Since M is even, by Lemma 1 with $R = T$, there is a perfect matching M_T of T such that $M_T \cup M$ is non-crossing. The perfect matching M_S is obtained similarly. $\square$

Corollary 17. *Let M be an even perfect matching. Let G be the visibility graph of $V(M)$ minus $E(M)$. Then G contains a graph-theoretic perfect matching (which possibly has crossings, but is the union of two non-crossing matchings).*

Note that the assumption that M is even is needed in Corollary 17— because of the instance in Figure 3 for example.

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References

[1] MANUEL ABELLANAS, ALFREDO GARCÍA, FERRAN HURTADO, JAVIER TEJEL, AND JORGE URRUTIA. Augmenting the connectivity of geometric graphs. *Comput. Geom. Theory Appl.*, 40(3):220–230, 2008.

[2] OSWIN AICHHOLZER, FRANZ AURENHAMMER, CLEMENS HUEMER, AND HANNES KRASSER. Transforming spanning trees and pseudo-triangulations. *Inform. Process. Lett.*, 97(1):19–22, 2006.

[3] OSWIN AICHHOLZER, FRANZ AURENHAMMER, AND FERRAN HURTADO. Sequences of spanning trees and a fixed tree theorem. *Comput. Geom. Theory Appl.*, 21(1-2):3–20, 2002.

[4] NADIA BENBERNOU, ERIK D. DEMAINE, MARTIN L. DEMAINE, MICHAEL HOFFMANN, MASHHOOD ISHAQUE, DIANE SOUVAINE, AND CSABA TÓTH. Disjoint segments have convex partitions with 2-edge connected dual graphs. In *Proc. 19th Canadian Conf. on Computational Geometry*, pp. 13–16, Carleton University, Ottawa, 2007.

[5] NADIA BENBERNOU, ERIK D. DEMAINE, MARTIN L. DEMAINE, MICHAEL HOFFMANN, MASHHOOD ISHAQUE, DIANE SOUVAINE, AND CSABA TÓTH. Erratum for "Disjoint segments have convex partitions with 2-edge connected dual graphs". In *Proc. 20th Canadian Conf. on Computational Geometry*, p. 223, McGill University, Montréal, 2008.

[6] TIBOR BISZTRICZKY AND GÁBOR FEJES TÓTH. A generalization of the Erdős-Szekeres convex n-gon theorem. *J. Reine Angew. Math.*, 395:167–170, 1989.

[7] TIBOR BISZTRICZKY AND GÁBOR FEJES TÓTH. Convexly independent sets. *Combinatorica*, 10(2):195–202, 1990.

[8] PROSENJIT BOSE, MICHAEL E. HOULE, AND GODFRIED T. TOUSSAINT. Every set of disjoint line segments admits a binary tree. *Discrete Comput. Geom.*, 26(3):387–410, 2001.

[9] KEVIN BUCHIN, ANDREAS RAZEN, TAKEAKI UNO, AND ULI WAGNER. Transforming spanning trees: A lower bound. In *Proc. 23rd European Workshop on Computational Geometry*, pp. 166–169, Graz University of Technology, Austria, 2007.

[10] ANDRÁS FRANK, TIBOR JORDÁN, AND ZOLTÁN SZIGETI. An orientation theorem with parity conditions. *Discrete Appl. Math.*, 115(1-3):37–47, 2001.

[11] ANDRÁS FRANK AND ZOLTÁN KIRÁLY. Graph orientations with edge-connection and parity constraints. *Combinatorica*, 22(1):47–70, 2002.

[12] MICHAEL HOFFMANN, BETTINA SPECKMANN, AND CSABA D. TÓTH. Pointed binary encompassing trees: Simple and optimal, 2007. *Proc. 14th Annual Fall Workshop on Computat. Geom.*, pp. 28–29, 2004. *Proc. 21st European Workshop on Comput. Geom.*, pp. 93–96, 2005. To appear in *Comput. Geom. Theory Appl.*

[13] MICHAEL HOFFMANN AND CSABA D. TÓTH. Alternating paths through disjoint line segments. *Inform. Process. Lett.*, 87(6):287–294, 2003.

[14] MICHAEL HOFFMANN AND CSABA D. TÓTH. Segment endpoint visibility graphs are Hamiltonian. *Comput. Geom. Theory Appl.*, 26(1):47–68, 2003.

[15] MICHAEL E. HOULE, FERRAN HURTADO, MARC NOY, AND EDUARDO RIVERA-CAMPO. Graphs of triangulations and perfect matchings. *Graphs Combin.*, 21(3):325–331, 2005.

[16] ANDRANIK MIRZAIAN. Hamiltonian triangulations and circumscribing polygons of disjoint line segments. *Comput. Geom. Theory Appl.*, 2(1):15–30, 1992.

[17] CRISPIN ST. J. A. NASH-WILLIAMS. Edge-disjoint spanning trees in finite graphs. *J. London Math. Soc.*, 36:445–450, 1961.

[18] JOSEPH O'ROURKE. *Art Gallery Theorems and Algorithms.* Clarendon Press, 1987.

[19] JÁNOS PACH AND EDUARDO RIVERA-CAMPO. On circumscribing polygons for line segments. *Comput. Geom. Theory Appl.*, 10(2):121–124, 1998.

[20] JÁNOS PACH AND JÓZSEF SOLYMOSI. Canonical theorems for convex sets. *Discrete Comput. Geom.*, 19(3):427–435, 1998.

[21] JÁNOS PACH AND GÉZA TÓTH. Erdős-Szekeres-type theorems for segments and noncrossing convex sets. *Geom. Dedicata*, 81(1-3):1–12, 2000.

[22] DAVID RAPPAPORT. Computing simple circuits from a set of line segments is NP-complete. *SIAM J. Comput.*, 18(6):1128–1139, 1989.

[23] DAVID RAPPAPORT, HIROSHI IMAI, AND GODFRIED T. TOUSSAINT. Computing simple circuits from a set of line segments. *Discrete Comput. Geom.*, 5(3):289–304, 1990.

[24] ANDREAS RAZEN. A lower bound for the transformation of compatible perfect matchings. In *Proc. 24th European Workshop on Computational Geometry*, pp. 115–118, 2008.

[25] CSABA D. TÓTH. Alternating paths along axis-parallel segments. *Graphs Combin.*, 22(4):527–543, 2006.

[26] GÉZA TÓTH. Finding convex sets in convex position. *Combinatorica*, 20(4):589–596, 2000.

[27] WILLIAM T. TUTTE. On the problem of decomposing a graph into n connected factors. *J. London Math. Soc.*, 36:221–230, 1961.

[28] MASATSUGU URABE AND MAMORU WATANABE. On a counterexample to a conjecture of Mirzaian. *Comput. Geom. Theory Appl.*, 2(1):51–53, 1992.