Tight Bounds for Visibility Matching of f-Equal Width Objects

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Abstract. Let s denote a compact convex object in $\mathbb{R}^2$. The *f-width* of s is the perpendicular distance between two distinct parallel lines of support of s with direction f. A set of disjoint convex compact objects in $\mathbb{R}^2$ is of equal f-width if there exists a direction f such that every pair of objects have equal f-width. A visibility matching, for a set of equal f-width objects is a matching using non-crossing lines of site in the visibility graph of the set. In this note we establish tight bounds on the size of a maximal visibility matching for a set of f-equal width objects by showing that $\left\lfloor \frac{2n}{3} \right\rfloor \leq h(n) \leq \frac{2n}{3}$.

1 Introduction

Let S denote a set of non-intersecting compact convex objects in $\mathbb{R}^2$. We say that two objects a and b from the set S *see each other* if there exists a straight line segment l with one point in a and one point in b such that l lies in the complement of S - $\{a, b\}$. We call such a line segment a *line of sight*. The *visibility graph* of S, denoted by $\mathrm{Vis}(S)$, associates a vertex to each object of S, and an edge between two vertices if and only if the associated objects see each other.

The combinatorial structure of the visibility graph for sets in $\mathbb{R}^2$ has been studied extensively. Some results on the combinatorial structure of the visibility graph of line segments can be found in [1, 2, 4, 6, 8].

In [4] the notion of set of f-equal width objects is introduced. Let s denote a compact object in $\mathbb{R}^2$. The *f-width* of s is the perpendicular distance between two distinct parallel lines of support of s with direction f. A set of compact objects in $\mathbb{R}^2$ is of *f-equal width* if there exists a direction f such that every pair of objects have f-equal width. A *visibility matching*, for a set of f-equal width objects is a matching using non-crossing lines of sight in the visibility graph of the set. Hosono [3] has shown that size of a maximal visibility matching of $2n$ f-equal width objects $h(n)$ satisfies the inequalities $\left\lfloor \frac{2n}{3} \right\rfloor \leq h(n) \leq \left\lfloor \frac{4n}{5} \right\rfloor$. In this note we establish tight bounds on the size of a maximal visibility matching for a set of f-equal width objects by showing that $\left\lfloor \frac{2n}{3} \right\rfloor \leq h(n) \leq \frac{2n}{3}$.

Some related results regarding the visibility graph of disjoint convex objects appear in a paper by Hosono, Meijer, and Rappaport [5] where it is shown that a set of translates of disjoint convex bodies admit a Hamilton path. In [7] it shown that the visibility graph of a set of disjoint congruent discs is Hamiltonian.
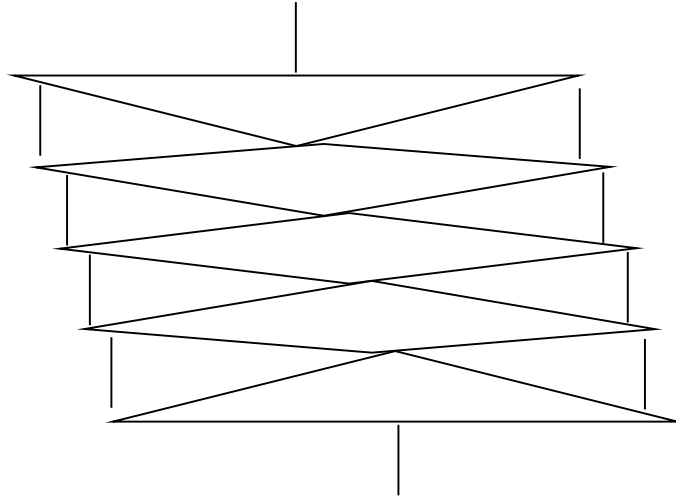
Fig. 1. This example leads to a maximal matching of at most $\frac{2n}{3}$.

2   Lower Bound

Throughout we assume that S is a set of f-equal width objects and f is horizontal.

Let Γ denote an arbitrary compact subset of $\mathbb{R}^2$. We say that a line l *supports* Γ if $l \cap \Gamma \neq \emptyset$, and a closed halfplane bounded by l contains Γ. If a point $\gamma \in \Gamma$ is contained in l then we say that γ is *extreme*.

For an element $s \in S$, let NORTH(s) be used to denote a horizontal line of support passing through an extreme point in s with maximum y coordinate, and let SOUTH(s) be used to denote a horizontal line of support passing through an extreme point in s with minimum y coordinate. Let $s, t \in S$, then $s < t$ if NORTH(s) is above NORTH(t), or when s and t are supported by the same horizontal line, then s is to the left of t. We use $<_{min}(S)$ to denote the least element of S using the $<$ ordering.

Consider a set S of f-equal width objects. We can partition the set S based on the sweep of a horizontal line . Let S_0 denote the subset of S intersected SOUTH$(<_{min}(S))$. Let $\Sigma_i = S - S_0 \cup \ldots \cup S_{i-1}$. We give a recursive definition for S_i the subset of S intersected by SOUTH$(<_{min}(\Sigma_i))$.

Observe that if the cardinality of the smallest S_i is 2 or more then we obtain our lower bound by simply matching objects within the same subset of our partition. It remains to settle the issue when there are subsets of cardinality 1.

Let $S_{j-1}, j \leq n - 1$ be a subset of cardinality 1, whose only element is called s. If S_j is of cardinality 1 then there is a line of sight between the element in S_j and s and they can be matched. Furthermore, if the cardinality of S_j is 2 or 4 or more then the matched objects in S_j overcome the unmatched s and the bound

is achieved. Thus consider the case where S_j is of cardinality 3, and let a left to right labelling of the objects of S_j be a, b, c. If s sees a or c we match all of the objects in the two sets. For example s sees a and since there is a horizontal line intersecting a, b, c and no other element of Σ_j then b must see c. Suppose on the other hand that s does not see either a or c. There is a horizontal line that intersects s and no other elements of Σ_{j-1} so s must see at least one object in S_j namely b. However, this implies that b is higher than both a or c (otherwise s could see a or c) so there is a horizontal line of sight between a and c. Therefore we match all of the objects in the union of S_{j-1} and S_j. Finally we have the case where S_{n-1} is of cardinality 1. In this event the best lower bound we can guarantee is $2n/3 - 1$.

3   Upper Bound

We give a construction that achieves the stated upper bound. Consider $3N$ f-equal width objects, as shown in Figure 1. The objects are a collection of $2N$ vertical unit length line segments called *bars* and N *gems*. Each gem is built upon a skeleton made up of a unit length vertical line segment and a horizontal line segment. See Figure 2.

Let $n = N - 1$ We can number the gems $g_0, \ldots, g_n$ from top to bottom. The horizontal segment of g_i is located a distance of i/n from the top of the vertical segment. Let c be a positive constant. Each horizontal segment of the gems skeleton is of length $2cn$ and is bisected by the vertical. The gems are placed one below the other, such that g_i is slightly to the right of g_{i-1}, and pushed up so that the gems are almost touching. That is there exists small positive values ϵ and $\epsilon_0, \epsilon_1, \ldots, \epsilon_n$ so that the top of g_i is positioned at coordinate $(i\epsilon, -i + \epsilon_i)$. We put a bar above the top gem and a bar below the bottom gem, and respectively call them top and bot. We can now fit the remaining $2n$ bars between the gems on the left and on the right. We refer to these bars as $l_1 \ldots l_n$ and $r_1 \ldots r_n$. There exists a positive value δ such that l_i can be placed directly above the left end of g_i so that the vertical gap between the segment and the boundaries of g_{i-1} and g_i is at least δ. On the right we place r_i directly below the right extreme of g_{i-1} again maintaining a gap of at least δ between the bar and its neighbouring gems. We prove that this construction is possible without overlapping any of the objects.
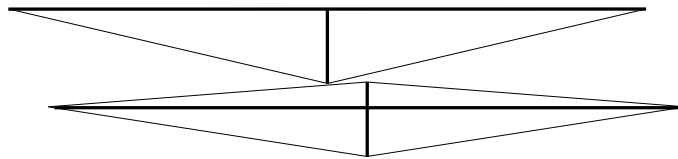
Fig. 2. The anatomy of a gem with its skeleton in bold lines.

Lemma 1. *The vertical distance between the left extreme of g_i and the boundary of g_{i-1} is at least $1 + 2\delta$*

Proof: Consider the distance between the two consecutive horizontal segments, say between g_{i-1} and g_i. This distance is at least $1 + 1/n - \epsilon_i$. Using similar triangles observe that the vertical distance of the boundary of g_{i-1} from the horizontal line segment is at most

$$\frac{\epsilon(1 - \frac{i}{n})}{cn} \leq \frac{\epsilon}{cn}.$$

Also observe that

$$\epsilon_i < \frac{\epsilon}{cn}, i = 0, \ldots n.$$

Therefore the vertical distance between the boundary of g_{i-1} and the left extreme of g_i is at least $1 + \frac{1}{n} - \frac{2\epsilon}{cn}$. If we set c to 4ϵ we have the desired result, then δ becomes $1/4n$.   □

We claim that the visibility graph of such a set of objects is isomorphic to a graph G with vertex set $\{g_0, g_1, \ldots, g_n, l_1, l_2, \ldots, l_n, r_1, r_2, \ldots, r_n, \text{top}, \text{bot}, \}$, and edges $\{(l_i, g_i), (l_i, g_{i-1}), (r_i, g_i), (r_i, g_{i-1}), (g_i, g_{i-1}), (\text{top}, g_0), (\text{bot}, g_n)\}$, for $i = 1 \ldots n$. See Figure 3.
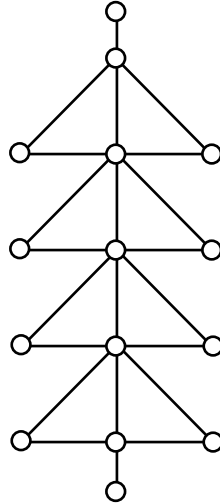

Fig. 3. The visibility graph for the objects.

Theorem 1. $\text{Vis}(S)$ *is isomorphic to G.*

Proof: The inclusion of the prescribed edges is not in question and can easily be verified. We show that no other edges exist in the visibility graph. It is

clear that top only sees g_0 and bot only sees g_n. Observe that the right and left extremes of the gems are on a straight line (they could also be put on the arc of a parabola, to avoid using degeneracies in our argument) it is clear the g_i sees g_{i-1} and g_{i+1} and no other gems. Similarly the bars r_i see no other r_j and no l_j, for $i \neq j$. Furthermore we claim that l_i cannot see r_i. The gap between g_i and g_{i+1} can be made as small as one wishes. Thus the only lines of sight that leak through the space between g_i and g_{i+1} are arbitrarily close to the boundaries of the respective gems. To be specific, the distance between gems effects the angle between the sight leak and the boundary of g_i. Thus we can make this angle as small as we like. On the other hand we also have control over the size of δ which is guaranteed to be some positive value. See Figure 4.
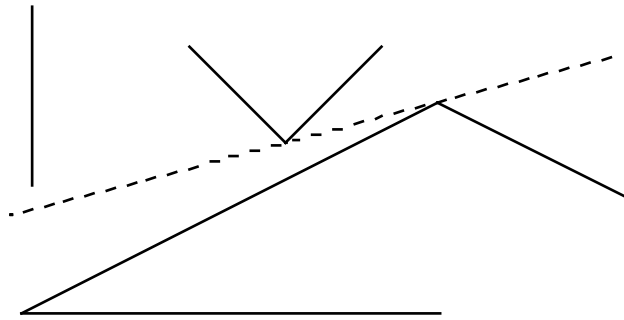
Fig. 4. Detail of a pair of gems and the amount of sight leak between them.

Since the bars are of distance at least δ from the gems there is no line of sight between any r_i and l_i. Thus we have shown that $\mathrm{Vis}(S)$ is isomorphic to G. □

It is now a routine matter to verify that a maximal matching in the derived visibility graph saturates $2N$ of the $3N$ objects.

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