Direct Planar Tree Transformation and Counterexample

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Abstract

We consider the problem of planar spanning tree transformation in a two-dimensional plane. Given two planar trees T' and T'' drawn on a set S of n points in general position in the plane, the problem is to transform T' into T'' by a sequence of simple changes called edge-flips or just flips. A flip is an operation by which one edge e of a geometric object is removed and an edge f ($f \neq e$) is inserted such that the resulting object belongs to the same class as the original object. Generally, for geometric transformation, the usual technique is to rely on some 'canonical' object which can be obtained by making simple changes to the initial object and then doing the reverse operations that transform the canonical object to the desired object. In this paper, we present a technique for such transformation that does not rely on any canonical tree. It is shown that T' and T'' can be transformed into each other by at most $n - 1 + k$ flips ($k \geq 0$) when S is in convex position and we also show results when S is in general position. We provide cases where the approach performs an optimal number of flips. A counterexample is given to show that if $|T' \setminus T''| = k$ then they cannot be transformed to one another by at most k flips.

1 Introduction

The problem of transforming of a certain class of geometric objects consisting of straight line segments and points in the plane, by applying small changes called flips in the objects, has been studied extensively [2, 5, 4, 6]. Given any two objects for a certain set of points, the question is whether the two objects can be transformed to each other by a sequence of flips and how many such transformations are required. A flip can be informally defined as the removal of an edge from, and insertion of another edge to, the object given. Originally, triangulations were investigated with positive results by K. Wagner [8]. Since then the problem has been studied for other classes of planar graphs such as tetrahedrons, linked-edge lists, pseudo-triangulations, planar spanning trees, crossing-free Hamiltonian paths and so on. Algorithms for such transformation as well as lower and upper bounds for achieving transformation results can be found in [1, 2, 4, 6].

One of the best-known results in the case of planar tree transformation is by Avis and Fukuda [2] who showed that for n points in general position every planar tree T' can be transformed into another planar tree T'' by means of at most $2n - 4$ flips. Later, the bound was slightly improved to $2n - m - s - 2$ in [7] (which is better if $m + s > 2$ otherwise it is at least as good as that in [2]). Here m is the maximum degree of any vertex v of T', where v is a point on the convex hull of the point set representing the vertex set and s is the degree of v in T''. Both of these approaches rely on the use of some 'canonical' tree. Informally, a canonical tree is a planar tree that has some particular characteristics such as, for example, all the vertices are directly connected to some vertex called the *root*. Surprisingly, most results related to transformations of different classes of graphs are based on the notion of some 'canonical' form of these graphs, as mentioned in [3]. The main idea of these techniques can be stated as follows: Given two objects A and B of a certain class of graph, the technique is to transform A into some canonical object C of that class by a sequence of transformations. Later, the sequence of transformations that transform B to C is reversed to obtain the desired transformation from A to B. This is an indirect approach. The main problem with this approach is that it takes a long sequence of additional flips to obtain the canonical graph even if the two objects are quite similar or they differ only in few edges.

Here we study the transformation of planar spanning trees using flips for a set S of n points in general position in the plane, avoiding the use of canonical trees. We provide results when the points are in convex position. With this approach, trees could be transformed in a more direct manner. We determine bounds on the number of transformations needed and show that an upper bound on the number of flips using this transformation is $n - 1 + k$, (k is the number of edges of one planar tree crossed by edges of the other planar tree drawn on S). We provide a counterexample where this direct approach cannot apply when S is in general position. Our algorithm obtains an optimal bound on the number of flips when there are no crossings.

The organization of the paper is as follows. In Section 2, we provide the definitions and terminologies that will be used throughout the paper. The technique of our algorithm for transformation and the results of the paper are presented in Section 3 and we conclude in Section 4.

2 Preliminaries

A **graph** $G = (V, E)$ consists of a set of vertices V, and an edge set $E = \{(v_i, v_j)|v_i, v_j \in V\}$. A graph G is called **planar** if it can be drawn in the plane so that no two edges cross, except at their common vertex. If $(v_i, v_j) \in E$, then v_i and v_j are **adjacent**. Let $S = \{v_0, v_1, v_2, \cdots, v_{n-1}\}$ be a set of n points in general position (no three points are collinear) in the two-dimensional Euclidean plane. Trees are drawn in the plane where the vertices (V) and edges (E) of a tree are represented by points of S and straight line-segments. Two vertices v_i and v_j, $v_i \neq v_j$ in an embedding of G are **visible** to each other if the straight line segment $(v_i, v_j) \in E$ between them does not cross any of the edges in G. A **flip** in tree T' is the operation of removal of an edge e and addition of an edge f such that $T'' = T' \backslash \{e\} \cup \{f\}$ is a tree.

Let $\mathcal{T}(S)$ denote the set of all trees of S and the geometric tree graph $T_G(S)$ denote the graph having $\mathcal{T}(S)$ as vertex set. Two trees $T', T'' \in \mathcal{T}(S)$ are adjacent if $T'' = T' \backslash \{e\} \cup \{f\}$ for some edges e and f. In the rest of the paper, it is assumed that a tree is planar unless otherwise mentioned.

3 Tree Transformation

Let $T' = (V, E')$ and $T'' = (V, E'')$ be any two trees belonging to $\mathcal{T}(S)$. It is required to construct T'' by applying a sequence of flips one by one to T'. In general, we say that T'' can be transformed from T' by p flips if there is a set of trees $T_0, T_1, \cdots, T_p$ where $T' = T_0$ and $T'' = T_p$ such that T_{i+1} can be obtained from T_i by a single flip. This implies that for any i, T_i and T_{i+1} are adjacent in $T_G(S)$ and it is known that the diameter of $T_G(S)$ is linear. Consider Fig. 1 where the tree T'' is obtained from T' by a sequence of flips.

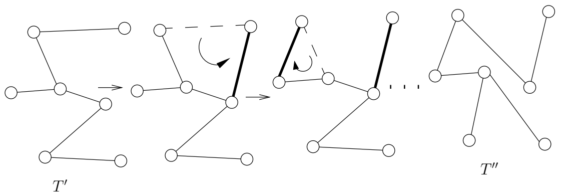


Figure 1: Transformations (shown with thick edges) applied on T' to construct T''.

In the following, we outline the main idea of our algorithm which does not rely on any form of canonical tree but obtains the desired transformation.

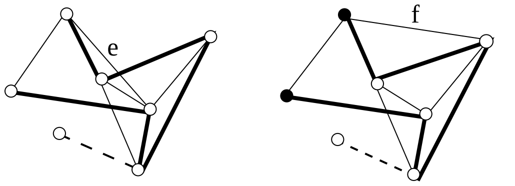
We draw two trees T' and T'' on S in the plane and obtain the graph $G = (V, E' \cup E'')$ where E' and E'' denote the edge-sets of T' and T'', respectively. Let $G_0 = G = (V, E_0 \cup E'')$ (where $E_0 = E'$). If there are p flips that transform T' into T'', our idea is to apply the sequence of flips on the edges of E' on G such that the resulting graphs are represented by $G_1 = (V, E_1 \cup E'')$, $G_2 = (V, E_2 \cup E'')$, $G_3 = (V, E_3 \cup E''), \cdots, G_p = T'' = (V, E'')$ where G_{i+1} is obtained from G_i by a single flip. In $G_i = (V, E_i \cup E'')$, E_i represents the edge set of $T_i = (V, E_i)$ being transformed into T''. Note that after the pth flip, the graph G_p turns into tree T'', since we expect that as flips are applied on the edges of T', gradually T' is turned into T'' and each instance of the intermediate trees $T_i = (V, E_i)$ along with $T'' = (V, E'')$ is reflected in G_p. In other words, we remember the order and the set of flips carried on G_i to produce G_p, then we apply these sequence of flips on T' in order to obtain T''.

To distinguish the edges of E_i from the edges of E'' in G_i, we color them with different colors. Edges $(u, v) \in E_i \backslash E''$ are colored in *red*, edges $(u', v') \in E'' \backslash E_i$ in *blue*, and edges $(u'', v'') \in E_i \cap E''$ in *purple*. Observe that, in graph G_i, only red edges can cross blue edges and there will be no crossings between red and purple or blue and purple edges since T' and T'' are planar. If a red edge is crossed by one or more blue edges, then we call it a *crossed red edge*. We count the total number of such crossed red edges after forming $G_0 = G = (V, E' \cup E'')$ at the beginning of our algorithm and denote it by k.

Lemma 1 *Suppose $G_i = (V, E_i \cup E'')$ is not planar. The removal of a crossed red edge, $e \in E_i \backslash E''$ from G_i splits E_i into two edge sets E'_i (and vertex set V'_i) and E''_i (and vertex set V''_i). Assume $CH(V) \cap V'_i \neq \emptyset$ and $CH(V) \cap V''_i \neq \emptyset$ where $CH(V)$ is the convex hull of V. There exists an edge $f \in V \times V$ such that $G_{i+1} = (V, E_i \backslash \{e\} \cup E'' \cup \{f\})$ where f is an edge on the convex hull of V connecting a vertex of V' to a vertex of V.*

Proof. Begin by removing a crossed red edge $e = (v_k, v_\ell)$ from $G_i = (V, E_i \cup E'')$ and obtain two edge sets E'_i and E''_i. Let $V'_i \subset V$ and $V''_i \subset V$ denote the incident vertices of E'_i and E''_i respectively. The aim is to connect $v_i \in V'_i$ and $v_j \in V''_i$ ($(v_i, v_j) = f$) so that the edge f does not cross any edges in G_{i+1}.

Color the vertices of V'_i and V''_i black and white, respectively. It suffices now to connect a black vertex to a white one without yielding any crossing. Select any of the black vertices v_i, $v_i \in V'_i$, on the convex hull of $CH(V)$ and start walking along the boundary of $CH(V)$ in some order. Once a walk is complete (that is, we reach the same vertex from which we started), we get a sequence of white and black vertices. We can insert an edge f by connecting any two consecutive white and black vertices in the sequence that does not generate any crossing in G_{i+1} since the edge is drawn on the boundary of $CH(V)$.



(a) $G_0 = (V, E' \cup E'')$ (b) $G_1 = (V, E_1 \cup E'')$

Figure 2: (a) Graph $G_0 = (V, E' \cup E'')$ is drawn with trees $T' = (V, E')$ and $T'' = (V, E'')$ where thin edges represent red edges, thick edges denote blue edges and dashed edges denote purple edges. (b) Edge e is removed and edge f is inserted without making any crossing in the graph. Black vertices belong to V_1 while the rest belong to V_2.

□

An illustration of the above lemma is shown in Fig.2.

We identify a case where we can obtain an optimal number of flips for the desired transformation: this is given in the following lemma.

Lemma 2 *Any tree $T' = (V, E')$ can be transformed into another tree $T'' = (V, E'')$ with at most $n-1$ flips when the number of crossed red edges is zero.*

Proof. Obtain the graph $G_0 = (V, E_0 \cup E'')$, where $E_0 = E'$. Since there are no crossed red edges, G_0 is planar. Begin in the following way. At each step, remove an arbitrary red edge $(u, v) \in E_i \backslash E''$ from G_i and colour the vertices black and white as in the proof of Lemma 1. Insert a purple edge between a black and a white vertex, otherwise the purple edge will make a cycle if the two incident vertices are of the same color. Since at every step a flip is carried out, we get a new graph $G_{i+1} = (V, E_{i+1} \cup E'')$, where $|E_{i+1} \cap E''| = |E_i \cap E''| + 1$ $(0 \leq i < p)$. The procedure stops when $|E_p \cap E''| = n-1$, meaning that T' has been transformed into T'' and G_p becomes $G_p = (V, E'')$. Since there can be zero purple edge in $G_0 = (V, E_0 \cup E'')$, the number of flips is at most $n-1$. □

The above two lemmas allow us to formulate the following theorem.

Theorem 3 *Any tree $T' = (V, E')$ can be transformed into another tree $T'' = (V, E'')$ with at most $n-1+k$ flips where k is the number of crossed red edges, provided that for any flip $1 \leq i \leq k$, $CH(V) \cap V_i' \neq \emptyset$ and $CH(V) \cap V_i'' \neq \emptyset$.*

Proof. Consider the graph $G_0 = (V, E_0 \cup E'')$, where $E_0 = E'$. The graph can be made planar by removing all the crossings between red and blue edges, as previously shown. Thus we need at most k flips to make the graph planar provided for any flip $1 \leq i \leq k$, $CH(V) \cap V_i' \neq \emptyset$ and $CH(V) \cap V_i'' \neq \emptyset$. As the graph is made planar, we can now follow Lemma 2 to obtain T''. It takes at most $n-1+k$ flips to transform T' into T''. □

If the set of points are in convex position, then each flip must reduce the number of crossings between red and blue edges by at least one, since there will always be two consecutive black and white points available to make the flip successful. Now we have the following corollary:

Corollary 4 *When the set of points is in convex position we need at most $n-1+k$ flips for the above transformation since for any flip $1 \leq i \leq k$, $CH(V) \cap V_i' \neq \emptyset$ and $CH(V) \cap V_i'' \neq \emptyset$.*

3.1 Counterexample

In this section, we show that there exist two trees defined on the same point set such that there does not exist any flip in one of the two trees that reduces the total number of crossings by at least 1 in G_i. Such an example is shown in Fig. 3 where the tree, T' in Fig. 3(a) has three edges different from the tree, T'' in Fig. 3(b), that is, $|T' \setminus T''| = 3$. However, there is no way (as evident from Fig. 3(c)) that any of the trees can be transformed to the other by three flips. This means that the direct transformation would fail after looking for all possible removal of edge crossings and this exhaustive searching would take time proportional to the number of crossings. However, once this fails we can then resort to the technique of using a canonical tree [1] which guarantees to take at most $2n-m-s-2$ flips for such transformation. The way the algorithm in [1] works is as follows: Let T' be the tree to be transformed into another tree T''. Let m be the maximum degree of any vertex v of T', where v is a point on the convex hull of the point set representing the vertex set. Similarly s is the degree of v in T''. We can first make T' into some canonical tree T_c where the degree of vertex v is $n-1$ by a sequence of $n-1-m$ flips and then perform the flips that transform T'' into T_c by having $n-1-s$ flips in reverse order. Thus we incur $2n-m-s-2$ flips for such transformation.

3.2 Remarks

The technique we present in this paper has the obvious advantage that in some cases it leads to the optimal number of flips to complete the transformation. It is well known that an approach for transforming a given tree (in general, it is true for other planar graphs of some class, e.g., planar paths, pseudotriangulations, etc.) into another via a flip operation which depends on a canonical tree never leads to the computation of

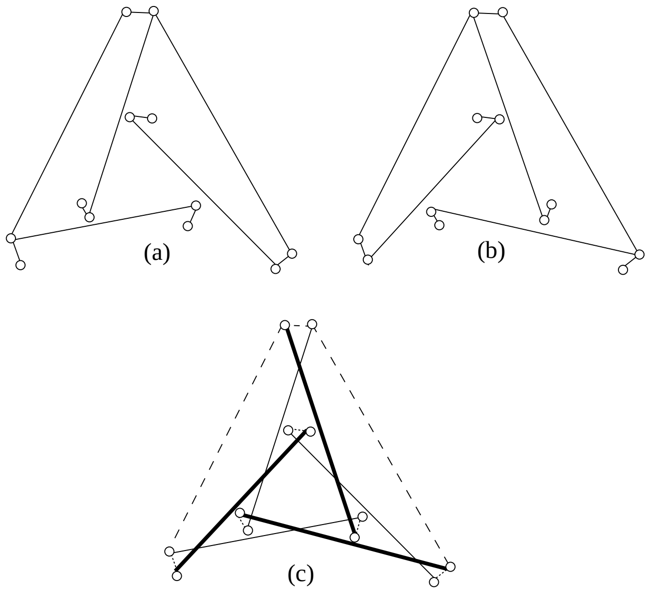


Figure 3: A counterexample with two trees defined on the same point where there does not exist any flip in one of the two trees such that the total number of crossings is reduced by at least one in G_i.

the optimal number of flips. This is because the two objects may differ only in a very small number of edges, whereas to transform them into a canonical form may take a large number of flips. As can be seen from Fig. 4, transforming any of the trees into the other takes 5 flips via a canonical tree based approach whereas only one flip suffices.

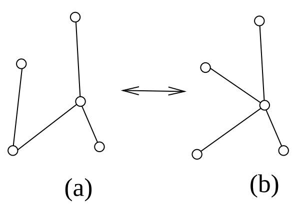


Figure 4: Transforming one tree into another takes only one flip optimally, but 5 flips through a canonical tree.

Finally, we provide a simple average case analysis of the number of flips of our algorithm. First, we determine the average number of crossed red edges. The number of crossed red edges varies from 0 to $n-3$. Thus, the total number of crossed red edges is $\sum_{k=0}^{n-3} k$ yielding the average number of crossed red edges, $\sum_{k=0}^{n-3} k/(n-2) = \frac{1}{n-2}(1+2+3+\cdots+n-3) = (n-3)/2$, where k is assumed to be uniformly distributed in $[0, n-3]$. Then the average number of flips required by our algorithm is $(n-1)+(n-3)/2 = 1.5n-2.5$. The above analysis is based on the fact that for any flip and for $1 \leq i \leq k$, $CH(V) \cap V_i' \neq \emptyset$ and $CH(V) \cap V_i'' \neq \emptyset$. However, the average-case analysis is based on the simplifying assumption that the number of crossings is uniformly distributed over a given interval. It is an interesting open problem to derive a more sophisticated value for the average number of flips required by our algorithm.

4 Conclusion

In this paper, we present a technique for tree transformation through flips when the points are in general position and also investigate the results when the points are in convex position. In this approach, we avoid the use of the canonical tree and directly transform one tree into another and show that it takes at most $n-1+k$ flips ($k \geq 0$) for such transformation when the points are in convex position. We also show results when the points are in general position and provide an upper bound on the number of flips. If there are no crossings in the union of edges of the given trees, it is shown that this technique performs an optimal number of flips. Finally, a counterexample is given to show that if two planar trees on the same point set differ by k edges, they cannot be transformed to one another by at most k flips.

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