Name: ______________________________ Student Number: ______________________	CISC 203 Discrete Mathematics for Computing Science Test 4, Fall 2010 Professor Mary McCollam

Please write in pen and only in the box marked "Answer".
This is a closed-book exam. No computers or calculators are allowed.

Question 1: [10 marks]

a) An employee joined a company in 2009 with a starting salary of \$50,000. Every year this employee receives a raise of \$1000 plus 5% of the salary of the previous year. Find a recurrence relation for the salary of this employee n years after 2009.

Answer:

Let a_n be the salary n years after 2009

$a_n = 1.05\, a_{n-1} + 1000$ with $a_0 = 50{,}000$

b) The following is the recurrence relation for a divide-and-conquer algorithm:

$$f(n) = 2\, f(n / 2) + 4$$

Use 'Theorem 1', given below, to construct a Big-Oh estimate for $f(n)$.
Show the intermediate steps.
Theorem 1: Let f be an increasing function that satisfies the recurrence relation $f(n) = a\, f(n / b) + c$, whenever n is divisible by b, where $a \geq 1$, b is an integer greater than 1, and c is a positive real number. Then $f(n)$ is:

$O(n^{\log_b a})$ if a > 1
$O(\log n)$ if $a = 1$

Answer:

$a = 2$ and $b = 2$

Since $a > 1$, we have $O(n^{\log_2 2}) = O(n)$

Question 2: [10 marks]

a) [8 marks] Determine whether the relation R on the set of all real numbers is reflexive, symmetric, asymmetric, and/or antisymmetric, where $(x,y) \in R$ if and only if $x = 2y$. Justify each answer with a brief explanation.

Answer:

Not reflexive, since it is not usually true that aRa, i.e., $x = 2x$. For instance, $1 \neq 2 \times 1$

Not symmetric, since it is not usually true that if aRb then bRa, i.e., if $x = 2y$ then $y = 2x$
For instance, $2 = 2 \times 1$, but $1 \neq 2 \times 2$

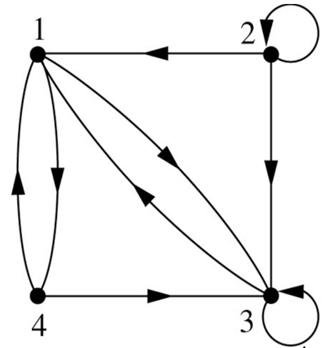
Not asymmetric, although it is usually true that if a is related to b, then b is not related to a, i.e., usually if $x = 2y$ then $y \neq 2x$.
However, if $a = 0$ and $b = 0$, then $a = 2b$ and $b = 2a$

R is **antisymmetric**, since if aRb and bRa, then $a = b$
Suppose $x = 2y$ and $y = 2x$.
Then $y = 4y$, from which it follows that $y = 0$ and hence $x = 0$.
Thus the only time when xRy and yRx is when $x = y$.

b) [2 marks] Let R be the relation on the set of people consisting of pairs (a,b), where a is a parent of b. Let S be the relation the set of people consisting of pairs (a,b) where a and b are siblings (brothers or sisters). What is $R \circ S$?

Answer:

a is an aunt or uncle of c



Question 3: [10 marks]

a) [4 marks] Determine whether the relation *R* represented by this directed graph is reflexive, symmetric, antisymmetric and/or transitive. Justify each answer with a brief explanation.

Answer:

Not reflexive, since there is not a loop at each vertex, e.g., there is no edge (1,1)

Not symmetric, since there is not always an edge (b,a) for every edge (a,b), e.g., there is an edge (2,1), but no edge (1,2)

Not antisymmetric, since there is sometimes both an edge (a,b) and edge (b,a) for distinct vertices a and b, e.g., there is both an edge (1,3) and an edge (3,1)

Not transitive, since for every edge (a,b) and edge (b,c), there is not always an edge (a,c). For instance, there is an edge (2,1) and an edge (1,4), but no edge (2,4).

b) [6 marks] Let R_1 and R_2 be relations on a set A = {1,2,3}.
R_1 = { (1,2), (2,1), (2,2), (2,3), (3,1) } and R_2 = { (1,2), (2,2), (2,3), (3,1), (3,2), (3,3) }.

Answer:

i) Represent each of these relations with a matrix with the elements of A listed in increasing order.

Matrix for R_1 contains:
$$\begin{matrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{matrix}$$

Matrix for R_2 contains:
$$\begin{matrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{matrix}$$

ii) Find the matrix that represents $R_1 \cap R_2$

The matrix of the intersection is formed by taking the meet:
$$\begin{matrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{matrix}$$

iii) Find the matrix that represents $R_2 \circ R_1$

The matrix is the Boolean product of the matrix for R_1 and the matrix for R_2:
$$\begin{matrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{matrix}$$

Question 4: [10 marks] Let ***R*** be the following relation on the set **{ x,y,z }**:

{ (x,x), (x,z), (y,y), (z,x), (z,y) }

Use the **0-1 matrix** representation for relations to find the transitive closure of ***R.*** Show the formula used to find the transitive closure of ***R*** from its 0-1 matrix representation and show the matrices in the intermediate steps in the algorithm, as well as the result matrix.

Answer:

The matrix for R, M_R, contains

1 0 1
0 1 0
1 1 0

The formula for the transitive closure of R is:

$$R^* = R \cup R^2 \cup R^3$$

The formula used to find the transitive closure of R from its 0-1 matrix is:

$$M_{R^*} = M_R \vee M_R^{[2]} \vee M_R^{[3]}$$

$M_R^{[2]}$ is the Boolean product of M_R and itself:

1 1 1
0 1 0
1 1 1

$M_R^{[3]}$ is the Boolean product of M_R and $M_R^{[2]}$:

1 1 1
0 1 0
1 1 1

$M_{R^*} = M_R \vee M_R^{[2]} \vee M_R^{[3]}$:

1 1 1
0 1 0
1 1 1

Therefore, R^* = { (x,x), (x,y), (x,z), (y,y), (z,x), (z,y), (z,z) }